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# Viscosity versus vorticity stretching: Global well-posedness for a family of Navier–Stokes-alpha-like models

Eric Olson[a,b,*], Edriss S. Titi[a,c,d]

[a] *Department of Mathematics, University of California, Irvine, CA 92697, USA*
[b] *Department of Mathematics, University of Nevada, Reno, NV 89557, USA*
[c] *Mechanical and Aerospace Engineering, University of California, Irvine, CA 92697, USA*
[d] *Department of Computer Science and Applied Mathematics, Weizmann Institute of Science, Rehovot 76100, Israel*

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**Abstract**

We study global well-posedness and regularity of solutions for a family of incompressible three-dimensional Navier–Stokes-alpha-like models that employ fractional Laplacian operators. This family of equations depends on two parameters, $\theta_1$ and $\theta_2$, which affect the strength of non-linearity (vorticity stretching) and the degree of viscous smoothing. Varying $\theta_1$ and $\theta_2$ interpolates between the incompressible Navier–Stokes equations and the incompressible (Lagrangian averaged) Navier–Stokes-$\alpha$ model. Our main result, which contains previously established results of J.L. Lions and others, provides a relationship between $\theta_1$ and $\theta_2$ that is sufficient to guarantee global existence, uniqueness and regularity of solutions.


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## 1. Introduction

Numerical solution of the Navier–Stokes equations for problems of engineering and geophysical relevance is not possible at present—even on the most powerful computers (see,

* Corresponding author at: Department of Mathematics, University of Nevada, Reno, Mail Stop 084, Reno, NV 89503, USA. Tel.: +1 775 784 6609.
*E-mail addresses:* ejolson@unr.edu (E. Olson), etiti@math.uci.edu, edriss.titi@weizmann.ac.il (E.S. Titi).

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e.g., [20,49]). Moreover, the mathematical theory for global existence and regularity of solutions to these equations is one of the most challenging open questions of mathematical analysis [17]. In turbulent fluid flows, most of the kinetic energy lies in the large scales, whereas the mathematical and computational difficulties lie in understanding the dynamical interaction between the significantly wide range of relevant scales in this multiscale phenomenon. To overcome this obstacle much effort is being made to produce reliable turbulence models which parameterize the effect of the active small scales in terms of the large scales.

Over the last years, the viscous Camassa–Holm equations have been proposed as a subgrid turbulence model: the (Lagrangian averaged) Navier–Stokes-$\alpha$ or LANS-$\alpha$ model. Physical derivations of the inviscid version of this model were proposed in Holm et al. [32], Holm [29] and in [10]. The inviscid version of this model also appears in the context of the non-Newtonian second grade fluids (see, for example, Dunn and Rajagopal [21], Joseph [36] and references therein). Adding an ad hoc viscous dissipation term, as proposed in [9–11,24], one reaches the viscous Camassa–Holm equations, also known as the LANS-$\alpha$ or Navier–Stokes-$\alpha$ model. This model was used in [9–11] as a closure model for the Reynolds averaged Navier–Stokes equations, and it was tested successfully against experimental measurements and direct numerical simulations of turbulent channel and pipe flows. Recently, it has been observed that an identical closure for the Reynolds equations may be obtained from the Leray-$\alpha$ model [14,35], the Clark-$\alpha$ model [8] and the Brandina model [15]. It is worth mentioning that one also obtains a similar set of reduced equations in channels and pipes when applying multipolar viscous fluid models (see, for example, Bellout et al. [5]). Cheskidov [13] and Holm et al. [34] used this model to obtain an extension of the Prandtl equations for the averaged flow in a turbulent boundary layer. Chen et al. [12] observed through direct numerical simulations that the energy spectra of the LANS-$\alpha$ model decay faster than $k^{-5/3}$ for wave numbers $k \gg 1/\alpha$. A more refined scaling argument in [23] indicates that the translational kinetic energy spectrum should scale as $\epsilon_\alpha^{2/3} k^{-5/3}(1 + \alpha^2 k^2)^{-2/3}$. Furthermore, Holm [30] showed that the LANS-$\alpha$ model possesses a Kármán–Howarth theorem consistent with these scalings. In addition, the LANS-$\alpha$ model enjoys a finite dimensional global attractor with an analytical upper bound on its fractal dimension that scales consistently with heuristic arguments for extensive three-dimensional turbulence, that is, as $(L/l_d)^3$. The history of the LANS-$\alpha$ model and information regarding its use as a subgrid scale turbulence model have been summarized by Holm et al. in [31]. Similar results have also been obtained for the Leray-$\alpha$ model [14] and the Clark-$\alpha$ model [8], with respective energy spectrum scalings of $(\epsilon_{\mathrm{Leray}})^{2/3} k^{-5/3}(1 + \alpha^2 k^2)^{-(n-6)/3}$ and $(\epsilon_{\mathrm{Clark}})^{2/3} k^{-5/3}(1 + \alpha^2 k^2)^{-(n-4)/3}$ where $n$ is an integer between 0 and 2 inclusive that indicates how the average velocity of an eddy of length $k^{-1}$ is computed. The Clark-$\alpha$ model also possesses a finite dimensional global attractor scaling as $(L/l_d)^3$, whereas the Leray-$\alpha$ model possess an attractor with bounds scaling as $(L/l_d)^{12/7}$, somewhere between one- and two-dimensional turbulence. Related models may be found in [15,35]. Note also the numerical work [27,28] of Holm and Guerts concerning the Leray-$\alpha$ model and the approximation of trajectory attractors of the Navier–Stokes equations as $\alpha \to 0$ in [57].

Not only does the LANS-$\alpha$ model have useful computational properties, but, as shown in [24], the mathematical theory of global existence and uniqueness of solutions for this model is complete. In this paper we consider a family of equations which interpolates between the Navier–Stokes equations and the LANS-$\alpha$ model and look for the limiting cases where we can prove global existence and uniqueness of regular solutions.

Let $\theta_1$ and $\theta_2$ be two non-negative parameters. The family of equations we shall consider have the form