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# On the stationary transport equations

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**Abstract**

We find solutions in $L^1$ and $L^\infty$ for the general domains for certain transport equations arising in the theory of the compressible Navier–Stokes equations. Some a priori estimates in $L^1$ and $L^\infty$ have been found. Then, the asymptotic behavior of $L^\infty$ solutions is found. We also consider solutions in $\mathcal{H}^1$ and BMO. We show the uniqueness of the solutions in these four function spaces.


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*Keywords:* Transport equations; Existence; $L^1$; $L^\infty$; $\mathcal{H}^1$; BMO; Uniqueness; Asymptotic behavior

## 1. Introduction

In this paper, we study the transport equations that describe transport phenomena such as heat transfer, mass transfer, fluid dynamics, etc. These expressions arise especially in the compressible Navier–Stokes equations and conservation laws. We refer the reader to [2,5] for the physical motivations, and to [4] for nonstationary transport equations.

We are interested in the existence of Lebesgue solutions under critical integrability conditions such as $L^1$ or $L^\infty$ in the general domains. We also develop an a priori norm estimate on the Hardy space $\mathcal{H}^1$ and BMO.

Let $\Omega$ be an open subset of $\mathbb{R}^n$ with smooth boundary. We denote by $L^p$ the Banach space $L^p(\Omega)$, $1 \le p \le \infty$, endowed with the norm $\|\cdot\|_{L^p}$, $1 \le p \le \infty$, and $W^{k,p}$ the Sobolev space $W^{k,p}(\Omega)$, endowed with the usual norm $\|\cdot\|_{W^{k,p}}$. Moreover $W_0^{k,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{k,p}$.

We consider the transport equation

$$\lambda\rho + (\mathbf{u}\cdot\nabla)\rho + a\rho = f, \tag{1.1}$$

where $\lambda$ is a positive constant, $a, f, \rho : \Omega \to R$ are scalar valued functions and $\mathbf{u} : \Omega \to \mathbb{R}^n$ is a vector valued

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function. We say $\rho \in L^1_{\mathrm{loc}}(\Omega)$ is a solution of (1.1) if

$$\int_\Omega \rho\,(\lambda\phi - \mathrm{div}(\mathbf{u}\phi) + a\phi)\,\mathrm{d}x = \int_\Omega f\phi\mathrm{d}x \tag{1.2}$$

for all $\phi \in C_0^\infty(\Omega)$.

Existence theorems in Sobolev spaces were previously established in the Hilbert space cases by Friedrichs [7], Lax and Phillips [9] and Kohn and Nirenberg [8]. Also for general $p \in (1,\infty)$ Fichera [6] and Oleinik and Radkevic [12] considered similar problems. Beirão da Veiga [1] proved the existence for the bounded domain in the context of $W^{k,p}$ theory for $p \in (1,\infty), k \geq -1$. Then, Novotný and Padula extended the existence results to exterior domains (see the appendix of [11]). For unbounded domain theory in $W^{k,p}$ space, we refer the reader to the paper of Novotný [10] when $k \geq 0$, $1 < p < \infty$, $u \in C^{k+1}(\overline{\Omega})$, $a \in C^k(\overline{\Omega})$.

Throughout the paper, we suppose that $\mathbf{u} \in W^{1,\infty}(\Omega)$ and $a \in L^\infty(\Omega)$ satisfy

$$\|a\|_{L^\infty} + \|\mathbf{u}\|_{W^{1,\infty}} \leq \frac{1}{2}\lambda. \tag{1.3}$$

The parameter $\lambda$ arises in the time discretization of the conservation law related to the CFL (Courant–Friedrichs–Lewy) condition. This assumption can be naturally extended according to the given situations. Moreover, we also assume that

$$\mathbf{u}(x)\cdot\nu(x) = 0 \quad \text{for all } x \in \partial\Omega \tag{1.4}$$

where $\nu(x)$ is the outward unit normal vector on $x \in \partial\Omega$. Indeed the problem will be well-posed if we assign boundary data on the set $\{x \in \partial\Omega : \mathbf{u}\cdot\nu < 0\}$ (see [5]). When $\mathbf{u}$ represents the velocity of the fluid, $\mathbf{u}$ will be zero on the boundary if the fluid is viscous flow. The following are the main results.

*Main results*

1. $L^\infty$ *Result*: If $f \in L^\infty$, then there is a unique solution $\rho \in L^\infty$ of the transport equations (1.1) which satisfies

   $$\|\rho\|_{L^\infty} \leq c\|f\|_{L^\infty}$$

   for some $c > 0$.
2. $L^1$ *Result*: If $f \in L^1$, then there is a unique solution $\rho \in L^1$ of the transport equations (1.1) which satisfies

   $$\|\rho\|_{L^1} \leq c\|f\|_{L^1}$$

   for some $c > 0$.
3. $\mathcal{H}^1$ *Result*: Assume $a = \mathrm{div}\,\mathbf{u}$. If $f \in \mathcal{H}^1$, then there is a unique solution $\rho \in \mathcal{H}^1$ of the transport equations (1.1) which satisfies

   $$\|\rho\|_{\mathcal{H}^1} \leq c\|f\|_{\mathcal{H}^1}$$

   for some $c > 0$.
4. BMO *Result*: Assume $a = 0$. If $f \in \mathrm{BMO}$, then there is a unique solution $\rho \in \mathrm{BMO}$ of the transport equations (1.1) which satisfies

   $$\|\rho\|_{\mathrm{BMO}} \leq c\|f\|_{\mathrm{BMO}}$$

   for some $c > 0$.

From the interpolation theorem, when the domain is bounded, we find an $L^p$ solution for given $f \in L^p$. This has already been proven by Beirão da Veiga [1] in the context of $W^{k,p}$ theory for $1 < p < \infty$ when $k \geq -1$, $\mathbf{u} \in W^{k+3,p}$, $a \in W^{k+2,p}$. We can also naturally extend the theorem for the derivatives, that is, if $a \in W^{k,p}$ and $\mathbf{u} \in W^{k+1,p}$ with sufficiently small norm as in Theorem 2.1, we can find a $W^{k,p}$ solution for all $1 \leq p \leq \infty$.

We denote a constant depending only on exterior data by $c$ and let $B_R(x_0)$ be the ball with radius $R$ centered at $x_0$. If there is no confusion, we often omit the center $x_0$ in various expressions.

In Section 2 we show the existence of solutions in $L^\infty$, $L^1$ spaces, and obtain the asymptotic behavior. In Section 3, we show the existence of solutions in $\mathcal{H}^1$, BMO spaces, and in Section 4 we provide the uniqueness of the solutions in $L^\infty$, $L^1$, $\mathcal{H}^1$, BMO spaces.