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Triple positive pseudo-symmetric solutions of three-point BVPs for p-Laplacian dynamic equations on time scales

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Abstract

Let $\mathbb{T}$ be a time scale such that $0, T \in \mathbb{T}$. We consider the three-point boundary value problem for p-Laplacian dynamic equations on time scales $\mathbb{T}$ of the form $(\phi_p(u^{\Delta}(t)))^{\nabla} + h(t)f(t, u(t)) = 0$ for $t \in (0, T)_{\mathbb{T}}$ with boundary conditions $u(0) = 0$, $u(\eta) = u(T)$, where $\mathbb{T}$ is symmetric in $[\eta, T]_{\mathbb{T}}$ and $\phi_p(u) = |u|^{p-2}u$ with $p > 1$. By using a pseudo-symmetric technique and the five-functionals fixed-point theorem, we prove that the boundary value problem has at least three positive pseudo-symmetric solutions under some assumptions. As an application, an example is given to illustrate the result.

MSC: 34B15; 39A10

Keywords: Time scales; Boundary value problem; Positive pseudo-symmetric solutions; p-Laplacian; Fixed-point theorem

1. Introduction

The theory of dynamic equations on time scales was introduced by Stefan Hilger in his Ph.D. thesis in 1988 [14]. In the past few years, it has found a considerable amount of interest and attracted the attention of many researchers. It is still a new subject, and research in this area is rapidly growing. On the one hand, the time scales approach not only unifies calculus and difference equations, but also solves other problems that have a mix of stop–start and continuous behavior. On the other hand, the time scales calculus has tremendous potential for application; for example, it can model insect populations that are continuous while in season and may follow a difference scheme with variable step size, die out in winter while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population [10]. By using the theory of time scales we can also study biological, heat transfer, stock market and epidemic models [1,15,24,29], etc.

Recently, for the existence problems of positive solutions of boundary value problems on time scales, some authors have obtained many results; for details, see [4,9,10,19,20,27] and the references therein. However, very little work has been done on the existence of positive solutions of three-point boundary value problems for p-Laplacian dynamic

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equations on time scales [3,12,25,26,28]. Other related results on p-Laplacian differential or difference equations appear in [6,13,18,21–23,30].

For convenience, throughout this paper, we denote by $\phi_p(u)$ a p-Laplacian operator, i.e., $\phi_p(u) = |u|^{p-2}u$ for $p > 1$ and $(\phi_p)^{-1} = \phi_q$, where $1/p + 1/q = 1$. We make the blanket assumption that $0, T$ are points in $\mathbb{T}$; by an interval $(0, T)_{\mathbb{T}}$ we always mean $(0, T) \cap \mathbb{T}$. Other kinds of intervals are defined similarly.

We would like to mention the results of He [12], Su and Li [25] and Sun and Li [26].

In [12], He considered the following three-point boundary value problems with p-Laplacian:

$$(\phi_p(u^{\Delta}(t)))^{\nabla} + h(t)f(u(t)) = 0, \quad t \in [0, T]_{\mathbb{T}}, \tag{1.1}$$

satisfying the boundary conditions

$$u(0) - B_0(u^{\Delta}(\eta)) = 0, \qquad u^{\Delta}(T) = 0, \tag{1.2}$$

or

$$u^{\Delta}(0) = 0, \qquad u(T) + B_1(u^{\Delta}(\eta)) = 0, \tag{1.3}$$

where $\eta \in (0, \rho(T))_{\mathbb{T}}$, and developed the existence of at least *two* positive solutions of the boundary value problems (1.1) and (1.2) or (1.3) by using the double-fixed-point theorem due to Avery and Henderson [7].

Recently, Su and Li [25] further considered the dynamic equation (1.1) with the boundary conditions (1.2) or (1.3). By using the five-functionals fixed-point theorem in a cone [8], the authors proved that the boundary value problem (1.1) and (1.2) or (1.3) has at least *three* positive solutions under some assumptions.

For the m-point boundary value problem with p-Laplacian

$$(\phi_p(u^{\Delta}(t)))^{\nabla} + h(t)f(t, u(t)) = 0, \quad t \in (0, T)_{\mathbb{T}}, \tag{1.4}$$

$$u^{\Delta}(0) = 0, \qquad u(T) = \sum_{i=1}^{m-2} a_i u(\xi_i), \tag{1.5}$$

Sun and Li [26] considered the existence of *single* or *multiple* positive solutions by using the Krasnosel'skii fixed-point theorem [11], the double-fixed-point theorem due to Avery and Henderson [7] and the Leggett–Williams fixed-point theorem [17].

It is also noted that the above mentioned references [12,25,26,28] only considered the existence of positive solutions. However, they did not further provide characters of positive solutions, such as pseudo-symmetry. It is now natural to consider the existence of pseudo-symmetric positive solutions of p-Laplacian dynamic equations on time scales.

In this paper, we consider the p-Laplacian boundary value problem on time scales $\mathbb{T}$ of the form

$$\big(\phi_p(u^{\Delta}(t))\big)^{\nabla} + h(t)f(t, u(t)) = 0 \quad \text{for } t \in (0, T)_{\mathbb{T}}, \tag{1.6}$$

$$u(0) = 0 \quad \text{and} \quad u(\eta) = u(T), \tag{1.7}$$

where $\eta \in (0, T)_{\mathbb{T}}$ and $\mathbb{T}$ is symmetric in $[\eta, T]_{\mathbb{T}}$. By using a pseudo-symmetric technique [6] and the five-functionals fixed-point theorem in a cone [8], we establish the existence of at least *three* positive pseudo-symmetric solutions of the boundary value problem (1.6) and (1.7). Our results are even new for the special cases of difference equations and include the results of Avery and Henderson [6] for differential equations. As an application, an example is given to illustrate the result.

For convenience, we now give some pseudo-symmetric definitions.

Definition 1.1. For any $\eta, T \in \mathbb{T}$ with $\eta < T$, a time scale $\mathbb{T}$ is said to be pseudo-symmetric if $\mathbb{T}$ is symmetric over the interval $[\eta, T]_{\mathbb{T}}$. That is, for any given $t \in [\eta, T]_{\mathbb{T}}$, we have $T - t + \eta \in [\eta, T]_{\mathbb{T}}$.

We note that such a pseudo-symmetric time scale $\mathbb{T}$ exists. For example, let $\eta = 0.22$, $T = 1$ and

$$\mathbb{T} = \{0, 0.05, 0.11, 0.16\} \cup [0.22, 0.44] \cup \{0.5, 0.55, 0.61, 0.67, 0.72\} \cup [0.78, 1].$$