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Existence of periodic solutions to a p-Laplacian Liénard differential equation with a deviating argument[☆]

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Abstract

By means of Mawhin's continuation theorem, a kind of p-Laplacian Liénard differential equation with a deviating argument as follows:

$$(\phi_p(y'(t)))' = f(y(t))y'(t) + h(y(t)) + g(y(t-\tau(t))) + e(t)$$

is studied. A new result on the existence of periodic solutions is obtained. The interest is that the relation between the existence of periodic solutions and the deviating argument $\tau(t)$ is investigated. Meanwhile, the approaches used to estimate a priori bounds of periodic solutions are different from the corresponding ones in the known literature.

MR Subject Classification: 34B15; 34B13

Keywords: Periodic solution; Coincidence degree; p-Laplacian differential equation; Deviating argument

1. Introduction

As is well known, the existence of periodic solutions to some second-order differential equations has been extensively studied [1–4]. In recent years, some researchers have investigated the problem of the existence of periodic solution to some differential equations involving the p-Laplacian or diffusion terms; see [5–12] and the references therein. For example, by using Mawhin's continuation theorem, Cheung and Ren studied the existence of T-periodic solutions to a p-Laplacian Liénard equation with a deviating argument in [5] as follows:

$$(\varphi_p(x'(t)))' + f(x(t))x'(t) + g(x(t-\tau(t))) = e(t),$$

and two results (Theorem 3.1 and Theorem 3.2 in [5]) on the existence of periodic solutions were obtained. The conditions imposed on $f(x)$ and $g(x)$ were ones such as:

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[C_1] There is a constant $d > 0$ such that $ug(u)$ does not change sign for $|u| > d$.
[C_2] There is a constant $l > 0$ such that

$$|g(u_1) - g(u_2)| \leq l|u_1 - u_2| \quad \forall u_1, u_2 \in R.$$

[C_3] There is a constant $\sigma > 0$ such that $|f(s)| \geq \sigma, \forall s \in R$.

Clearly, condition [C_2] implies

$$\lim_{|u|\to+\infty} \frac{|g(u)|}{|u|} = \lim_{|u|\to+\infty} \frac{|g(u) - g(0) + g(0)|}{|u|} \leq l. \tag{1.1}$$

The aim of this paper is to study the existence of periodic solutions to a class of p-Laplacian Liénard equations with a deviating argument as follows:

$$(\varphi_p(y'(t)))' + f(y(t))y'(t) + h(y(t)) + g(y(t - \tau(t))) = e(t), \tag{1.2}$$

where $p > 1$ is a constant, $\varphi_p : R \to R, \varphi_p(u) = |u|^{p-2}u,\ f, g, e, \tau \in C(R, R)$ with $\tau(t + T) \equiv \tau(t)$ and $e(t+T) \equiv e(t)$, $T > 0$ is a given constant. Such equations are derived from many fields, such as fluid mechanics and nonlinear elastic mechanics. The significance of this paper is that the main result is related to the deviating argument $\tau(t)$, and the methods used to estimate *a priori bounds* of periodic solutions are different from the corresponding ones in [1–5,10]. Furthermore, even for the case of $h(x) \equiv 0$, the conditions imposed on $f(x)$ and $g(x)$ are different from the corresponding ones in [5]. For example, we only require that the function $f(x)$ is continuous on R, which is weaker than condition [C_3]; and also the growth condition imposed on $g(x)$ is

$$\lim_{|u|\to+\infty} \frac{|g(u)|}{|u|^{p-1}} \leq r \in [0, +\infty). \tag{1.3}$$

Obviously, if $p \geq 2$, one can find from (1.1) that condition (1.3) is weaker than condition [C_2].

2. Main lemmas

The following lemma is crucial for investigating the relation between the existence of periodic solutions to Eq. (1.2) and the deviating argument $\tau(t)$.

Lemma 2.1 (*[13]*). *Let $p \in (1, +\infty)$ be a constant, $s \in C(R, R)$ such that $s(t + T) \equiv s(t)$, $u \in C^1(R, R)$ with $u(t + T) \equiv u(t)$. Then*

$$\int_0^T |u(t) - u(t - s(t))|^p \mathrm{d}t \leq 2\left(\max_{t\in[0,T]} |s(t)|\right)^p \int_0^T |u'(t)|^p \mathrm{d}t.$$

Lemma 2.2 (*[14]*). *Let $s, \sigma \in C(R, R)$ with $s(t+T) \equiv s(t)$ and $\sigma(t+T) \equiv \sigma(t)$. Suppose that the function $t-\sigma(t)$ has a unique inverse $\mu(t), \forall t \in R$. Then $s(\mu(t + T)) \equiv s(\mu(t))$.*

Now, we recall Mawhin's continuation theorem which our study is based upon.

Let X and Y be real Banach spaces and $L : D(L) \subset X \to Y$ be a Fredholm operator with index 0; here $D(L)$ denotes the domain of L. This means that $\operatorname{Im} L$ is closed in Y and $\dim\ker L = \dim(Y/\operatorname{Im} L) < +\infty$. Consider the supplementary subspaces X_1 and Y_1 such that $X = \ker L \oplus X_1$ and $Y = \operatorname{Im} L \oplus Y_1$ and let $P : X \to \ker L$ and $Q : Y \to Y_1$ be the natural projections. Clearly, $\ker L \cap (D(L) \cap X_1) = \{0\}$; thus the restriction $L_P := L|_{D(L)\cap X_1}$ is invertible. Denote by K the inverse of L_P.

Now, let Ω be an open bounded subset of X with $D(L) \cap \Omega \neq \phi$. A map $N : \overline{\Omega} \to Y$ is said to be L-compact in $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and the operator $K(I - Q)N : \overline{\Omega} \to X$ is compact.

Lemma 2.3 (*[15]*). *Suppose that X and Y are two Banach spaces, and $L : D(L) \subset X \to Y$ is a Fredholm operator with index 0. Furthermore, $\Omega \subset X$ is an open bounded set and $N : \overline{\Omega} \to Y$ is L-compact on $\overline{\Omega}$. If:*

(1) $Lx \neq \lambda Nx, \forall x \in \partial\Omega \cap D(L), \lambda \in (0, 1)$;