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An existence theorem for weak solutions for a class of elliptic partial differential systems in general Orlicz–Sobolev spaces

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Abstract

I prove the existence of a weak solution for the Dirichlet problem of a class of elliptic partial differential systems $-\frac{\partial A_\alpha^i}{\partial x^\alpha}(x, u(x), \mathrm{D}u(x)) + B^i(x, u(x), \mathrm{D}u(x)) = 0$ in general Orlicz–Sobolev spaces $W_0^1 L_M(\Omega, R^N)$, where $i = 1, \dots, N, \alpha = 1, \dots, n$, $u : \Omega \to R^N$ is a vector-valued function, and the summation convention is used throughout with i, j running from 1 to N and α, β running from 1 to n.

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1. Introduction

Let $\Omega \subset R^n$ be a bounded Lipschitz domain. Fu, Dong and Yan [4] study the weak solution of the Dirichlet problem

$$-\frac{\partial A_\alpha^i}{\partial x^\alpha}(x, u(x), \mathrm{D}u(x)) + B^i(x, u(x), \mathrm{D}u(x)) = 0, \quad x \in \Omega, \tag{1.1}$$

$$u^i(x) = 0, \quad x \in \partial\Omega, \tag{1.2}$$

in separably reflexive Orlicz–Sobolev spaces $W_0^1 L_M(\Omega, R^N)$, where $u : \Omega \to R^N$ is a vector-valued function, and the summation convention is used throughout with i, j running from 1 to N and α, β running from 1 to n; M and $\bar{M}$ are a pair of complementary N-functions.

A simple case of (1.1) is the Euler–Lagrange system.

Dong and Shi [3] prove the existence for the weak solutions of (1.1) and (1.2) in separable Orlicz–Sobolev spaces $W_0^1 L_M(\Omega, R^N)$, but the result is restricted to N-functions M satisfying a Δ_2-condition.

It is our purpose in this paper to extend the result of [3] to general N-functions (i.e. without assuming a Δ_2-condition on M).

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2. Preliminaries

Let $M : R \to R$ be an N-function, i.e. M is even, convex, $M(u)/u \to 0$ as $u \to 0$, and $M(u)/u \to \infty$ as $u \to \infty$. $\bar{M}$ is its complementary function. ϕ and ψ are their right-hand derivatives, respectively. The N-function M is said to satisfy the Δ_2 condition near infinity ($M \in \Delta_2$) if for some $K > 0$ and $\bar{u} > 0$,

$$M(2u) \le KM(u), \quad \forall u \ge \bar{u}. \tag{2.1}$$

Let P be an N-function. $P \ll M$ means that P grows essentially faster than M; that is for each $k > 0$, $P(kt)/M(t) \to 0$ as $t \to \infty$. This is the case if and only if $M^{-1}(t)/P^{-1}(t) \to 0$ as $t \to \infty$.

Let Ω be a bounded domain in R^n, $u : \Omega \to R^N$ be a vector-valued function. Its module is introduced by

$$\rho_M(u) = \int_\Omega M(|u(x)|)\mathrm{d}x.$$

The class $W^1L_M(\Omega, R^N)$ (resp., $W^1E_M(\Omega, R^N)$) consists of all functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega, R^N)$ (resp., $E_M(\Omega, R^N)$). The Orlicz space $L_M(\Omega, R^N)$ is endowed with the Luxemburg norm

$$\|u\|_{(M)} = \inf\left\{\lambda > 0; \int_\Omega M\left(\frac{|u(x)|}{\lambda}\right)\mathrm{d}x \le 1\right\},$$

or the Orlicz norm

$$\|u\|_M = \inf_{k>0} \frac{1}{k}\left\{1 + \int_\Omega M(k|u(x)|)\mathrm{d}x\right\}.$$

Then $\|u\|_{(M)} \le \|u\|_M \le 2\|u\|_{(M)}$ (e.g. inequality (9.24) in [7]). The class $W^1L_M(\Omega, R^N)$ of such functions may be given a norm

$$\|u\|_{M,\Omega} = \sum_{|\alpha|\le 1} \|D^\alpha u\|_{(M)}.$$

It is a Banach space under this norm, referred to as Orlicz–Sobolev space. Thus $W^1L_M(\Omega, R^N)$ and $W^1E_M(\Omega, R^N)$ can be identified with subspaces of the product of $n+1$ copies of $L_M(\Omega, R^N)$. Denoting this product by ΠL_M, we will use the weak topologies $\sigma(\Pi L_M, \Pi E_{\bar{M}})$ and $\sigma(\Pi L_M, \Pi L_{\bar{M}})$.

The space $W_0^1E_M(\Omega, R^N)$ is defined as the (norm) closure of $C_0^\infty(\Omega, R^N)$ in $W^1E_M(\Omega, R^N)$ and the space $W_0^1L_M(\Omega, R^N)$ as the $\sigma(\Pi L_M, \Pi E_{\bar{M}})$ closure of $C_0^\infty(\Omega, R^N)$ in $W^1L_M(\Omega, R^N)$.

We say that u_n converges to u for the modular convergence in $W^1L_M(\Omega)$ if for some $\lambda > 0$,

$$\int_\Omega M\left(\frac{|D^\alpha u_n - D^\alpha u|}{\lambda}\right) \to 0, \quad \forall|\alpha| \le 1. \tag{2.2}$$

Definition 2.1. A function $f : R^n \times R^N \times R^{Nn} \to R$ is called a Caratheodory function if it satisfies: $\forall(s, p) \in R^N \times R^{Nn}$, $x \to f(x, s, p)$ is measurable; for almost every $x \in R^n$, $(s, p) \to f(x, s, p)$ is continuous.

Lemma 2.1 (*Lieberman [10]*). *For all* $u \in W_0^1L_M(\Omega)$,

$$\int_\Omega M(K^*|u|)\mathrm{d}x \le \int_\Omega M(|\mathrm{D}u|)\mathrm{d}x$$

where $K^* = 1/\mathrm{diam}\ \Omega$ *and* diam Ω *is the diameter of* Ω.

Lemma 2.1 is referred to as Lemma 2.2 in [10].

Definition 2.2. Let $V_n = \mathrm{Span}\{\omega_1, \ldots, \omega_n\}$; then $u_n \in V_n$ is called a Galerkin solution of $A(u) = f$ in V_n if and only if

$$(A(u_n), v) = (f, v) \quad \forall v \in V_n.$$