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Existence and multiplicity of nodal solutions for Dirichlet problems in upper half strip with holes

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Abstract

In this paper, we consider the existence and multiplicity of nodal solutions of semilinear elliptic equations. We prove that a semilinear elliptic equation in large domains does not admit any least energy nodal (sign-changing) solution and in an upper half strip with m-holes has at least m^2 2-nodal solutions.

Keywords: Semilinear elliptic equations; Nodal solutions; Nehari manifold

1. Introduction

In this paper, we study the existence and multiplicity of nodal solutions of semilinear elliptic equations of the form

$$\begin{cases} -\Delta u + u = |u|^{p-2}u^+ + |u|^{q-2}u^- & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$E_{p,q}$}$$

where Ω is a domain in $\mathbb{R}^N$, $2 < p, q < \infty$ $(N = 2), 2 < p, q < \frac{2N}{N-2}$ $(N \geq 3), u^+ = \max\{0, u\}$ and $u^- = \min\{u, 0\}$. Associated with Eq. $(E_{p,q})$, we consider the energy functional J in the Sobolev space $H_0^1(\Omega)$,

$$J(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p}\int_\Omega |u^+|^p - \frac{1}{q}\int_\Omega |u^-|^q,$$

where $\|u\| = \left(\int_\Omega |\nabla u|^2 + u^2\right)^{1/2}$ is a standard norm in $H_0^1(\Omega)$. It is well-known that the functional $J \in C^2(H_0^1(\Omega), \mathbb{R})$ and the solutions of Eq. $(E_{p,q})$ in Ω are the critical points of the energy functional J in $H_0^1(\Omega)$.

Generally, a standard technique to find the one sign solutions of Eq. $(E_{p,q})$ in Ω is using the Nehari minimization problems:

$$\alpha^{\pm}(\Omega) = \inf_{v \in \mathbf{M}^{\pm}(\Omega)} J(v),$$

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where $\mathbf{M}^{\pm}(\Omega) = \{u \in H_0^1(\Omega) \setminus \{0\} \mid \langle J'(u), u\rangle = 0, \pm u \geq 0\}$. Note that $\alpha^{\pm}(\Omega)$ are positive numbers and $\alpha^{\pm}(\Omega_1) \geq \alpha^{\pm}(\Omega_2)$ if $\Omega_1 \subset \Omega_2$ (see Willem [18]). Furthermore, we called a nonzero critical point u_0 of J is a least energy positive (or negative) solution of Eq. $(E_{p,q})$ in Ω if $u_0 > 0$ (or <0) and $J(u_0) = \alpha^{+}(\Omega)$ (or $\alpha^{-}(\Omega)$).

By the Rellich compactness theorem, it is easy to obtain a least energy positive (or negative) solution of Eq. $(E_{p,q})$ in bounded domains. For general unbounded domains Ω, because of the lack of compactness, the existence of one sign solutions of Eq. $(E_{p,q})$ in Ω is very difficult and unclear. Indeed, a by now classical result of Esteban–Lions [11] states that for unbounded domains satisfying the condition: there exists $\chi \in \mathbb{R}^N$, $\|\chi\| = 1$ such that $n(x) \cdot \chi \geq 0$ and $n(x) \cdot \chi \not\equiv 0$ on $\partial\Omega$, where $n(x)$ is the unit outward normal vector to $\partial\Omega$ at the point x, Eq. $(E_{p,q})$ does not admit any nontrivial solution. Recently, there have been some progresses for the existence of least energy positive (or negative) solutions of Eq. $(E_{p,q})$ in unbounded domains as follows: Berestycki–Lions [5] for $\Omega = \mathbb{R}^N$, Lien–Tzeng–Wang [15] for Ω is a periodic domain, Del Pino–Felmer [9,10] for Ω is a quasicylindrical domain, Wu [20] for Ω is a multi-bump domain. On the other hand, when Ω is an exterior domain in $\mathbb{R}^N$, it is well-known that Eq. $(E_{p,q})$ in exterior domain does not admit any least energy positive (or negative) solution (see Benci–Cerami [4]). However, Benci–Cerami [4] proved that Eq. $(E_{p,q})$ in exterior domain has a higher energy positive solution.

In the aforementioned works, the authors considered one sign solutions. For other situations, Bartsch [2] obtained infinite nodal (sign-changing) solutions for Eq. $(E_{p,q})$ in bounded domains. Furtado [12,13], showed that the domain topology is related with the number of 2-nodal solutions of Eq. $(E_{p,q})$, where the definition of 2-nodal solution is: for a nontrivial solution u is such that the set $\{x \in \Omega \mid u(x) \neq 0\}$ has exactly two connected components, u is positive in one of them and negative in the other (see Castro–Clapp [7] or Bartsch–Weth [3]). Huang–Wu [14] proved that Eq. $(E_{p,q})$ in a finite strip with a hole has at least four 2-nodal solutions. Bartsch–Weth [3], proved that the Eq. $(E_{p,q})$ in a bounded domain Ω that contains a large ball having three nodal solutions in which two are the 2-nodal solutions. Wu [20], proved that the Eq. $(E_{p,q})$ in an m-bump domain has at least m^2 2-nodal solutions.

Motivated by the above results, we are interested in the relation between the topology of domain and the existence of nodal solutions of Eq. $(E_{p,q})$. Before stating our main results, we need the following definitions and notations. Denote the N-ball $B^N(z_0; r)$ in $\mathbb{R}^N$, the infinite strip $\mathbf{A}$, the upper half strip $\mathbf{A}^+$ and the finite strip $\mathbf{A}(s, l)$ as follows:

$$B^N(z_0; r) = \{z \in \mathbb{R}^N : |z - z_0| < r\};$$
$$\mathbf{A} = \{(x, y) \in \mathbb{R}^{N-1} \times \mathbb{R} : x \in \omega, \omega \text{ is a bounded domain in } \mathbb{R}^{N-1}\};$$
$$\mathbf{A}^+ = \{(x, y) \in \mathbf{A} : y > 0\};$$
$$\mathbf{A}(s, l) = \{(x, y) \in \mathbf{A} : s < y < l\}.$$

Definition 1.1. (i) The domain Ω is called large domain in $\mathbf{A}$ if $\Omega \subset \mathbf{A}$ and for any $n > 0$ there exists $s < l$ such that $l - s = n$ and $\mathbf{A}(s, l) \subset \Omega$;
(ii) The domain Ω is called strictly large domain in $\mathbf{A}$ if Ω is a large domain in $\mathbf{A}$ and $\Omega \neq \mathbf{A}$.

Note that the infinite strip $\mathbf{A}$ is a large domain in itself and the upper half strip with m-holes

$$\Omega(t) = \mathbf{A}^+ \setminus \left[\cup_{i=1}^{m} \overline{B^N((0, it); r_0)}\right]$$

is a strictly large domain in $\mathbf{A}$, where $t > 2r_0 > 0$ and $B^{N-1}(0; r_0) \subsetneq \omega$. Furthermore, Eq. $(E_{p,q})$ in $\mathbf{A}$ has a ground state solution and in $\Omega(t)$ does not admit any least energy positive (or negative) solution for all $t > 0$ (see Wu [19, Lemma 11]). Thus, Eq. $(E_{p,q})$ in $\Omega(t)$ only has higher energy solution. However, Wu [19] proved that Eq. $(E_{p,q})$ in $\Omega(t)$ has at least m higher energy positive solutions for t sufficiently large.

In this paper, we can show that Eq. $(E_{p,q})$ in large domains does not admit any least energy nodal solution. Here all nodal solutions of Eq. $(E_{p,q})$ lie in the set

$$\mathbf{N}(\Omega) = \left\{u \in H_0^1(\Omega) \mid u^{\pm} \in \mathbf{M}^{\pm}(\Omega)\right\}.$$

Let $\theta(\Omega) = \inf_{u \in \mathbf{N}(\Omega)} J(u)$. Then we have the following result.

Theorem 1.2. *If Ω is a large domain in $\mathbf{A}$, then Eq. $(E_{p,q})$ in Ω does not admit any nodal solution v_0 such that $J(v_0) = \theta(\Omega)$, that is Eq. $(E_{p,q})$ in Ω does not admit any least energy nodal solution. Furthermore, $\theta(\Omega) = \alpha^{+}(\mathbf{A}) + \alpha^{-}(\mathbf{A})$.*