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About some new integral inequalities of Wendroff type for discontinuous functions

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Abstract

In this article, we obtain some new nonlinear integral inequalities for discontinuous functions of two independent variables (Wendroff type) by including also inequalities with delay. We deduce new generalizations of earlier results given by R.P. Agarwal, R. Bellman, I. Bihari, B.K. Bondge, V. Lakshmikantham, S. Leela, B.G. Pachpatte for continuous and discrete functions. Furthermore, generalizations of some results for integro-sum inequalities are obtained as well.

MSC: 26D10; 26D15; 26D20

Keywords: Integral inequalities; Discontinuous functions; Integro-sum inequalities

1. Introduction

Our article is devoted to the development of the integral inequalities method as a tool for investigating the qualitative characteristic solutions of different kinds of equations: differential equations, difference equations, partial differential equations, impulsive differential equations [1, 6,19–22,25,27,36,37]. By using results given by Agarwal [1,2], Agarwal and Leela [3], Akinyele [4], Bainov and Simeneov [6], Bellman [7], Bihari [8], Bondge and Pachpatte [9–11],

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Gronwall [23], Lakshamikantham [24], Lakshamikantham and Leela [25], Lakshamikantham et al. [26], Lakshamikantham et al. [27], Pachpatte [28–31], Shih and Yeh [39], Singare and Pachpatte [40], Snow [41,42], Thandapani and Agarwal [43], Walter [44], Willet and Wong [45], Yeh [46,47,49], Yeh and Shih [48], Young [50], Zahariev and Bainov [51] for continuous and discrete generalization of the Gronwall–Bellman–Bihari type inequalities and results given by Azbelev and Tsalyuk [5], Rakhmatullina [32] for investigating the conditions of the solvability of Chaplygin's problem for the integral equation (inequality) of Volterra type (one-dimensional, continuous case), we consider new integral inequalities for discontinuous functions of Wendroff type. From our results we also deduce new generalizations of previous results given by Borysenko [12,14,17], by Borysenko [18], by Samoilenko and Borysenko [33,34] for integro-sum inequalities. It is useful to note that in the earlier articles [12,13,33], integro-sum inequalities for the piecewise-continuous functions of a certain type:

$$\varphi(t) \leq \psi(t) + \int_{t_0}^{t} K(t, s, \varphi(s))\mathrm{d}s + \sum_{t_0 < t_i < t} \mu(t, t_i)\sigma_k(\varphi(t_i - 0)),$$

were investigated; here $\varphi(t)$, $\psi(t)$, $\mu(t, t_i)$, $\sigma_i(u)$ are continuous nonnegative functions ($i = 1, 2, \ldots$) for $t \geq t_0 \geq 0$, except for $\varphi(t)$, which has the first kind of discontinuities at the points $\{t_k\}$, $k = \overline{1, \infty} : 0 \leq t_0 < t_1 < \ldots$, $\lim_{i\to\infty} t_i = \infty$.

The kernel $K(t, s, u)$, which is nonnegative at $t \geq s \geq t_0$, is determined in domain $t \geq s \geq t_0$, $|u| \leq k = \text{const} > 0$, and at fixed t and s it is nondecreasing with respect to u; the functions $\sigma_k(u)$ are continuous nonnegative and nondecreasing with respect to u.

In the work [14] the first author obtained a certain estimate:

$$\varphi(t) \leq \xi_\psi(t), \quad \forall t \in [0, +\infty],$$

where $\xi_\psi(t)$ is some solution of the integro-sum equation:

$$\xi(t) = \psi(t) + \int_{t_0}^{t} K(t, s, \xi(s))\mathrm{d}s + \sum_{t_0 < t_i < t} \mu(t, t_i)\sigma_i(\xi(t_i - 0))$$

continuous in each of the intervals $[t_i, t_{i+1}[$, $i = 0, 1, \ldots$, and has some first kind of discontinuities at the points $\{t_i\}$.

From the result [14], the results in [12,38] follow for integro-sum inequalities.

In all the previously described results for integro-sum inequalities before the results in [18] (see [12,13,15,16,19–22,33,35,37,38]) there were considered Lipschitz type nonlinearities for $\sigma_k(u)$. In the works [14,17,37] there were obtained generalization integral inequalities of Bellman–Bihari type for functions of two independent variables with jumps (finite) at some fixed points from an open domain $\Omega \subset \mathbb{R}^2_+$.

In this article, in a similar way to [18] for the one-dimensional case, we investigate the integral inequalities for the functions of two independent variables (Wendroff type) with non-Lipschitz type functions, which characterize the values of discontinuity (the results [14,17,37] follow automatically as particular cases of the results of this article).

In the Section 2, we obtain a new analogy of the results by Bellman and Bihari for discontinuous functions (two-dimensional case).

In Section 3, we generalize the results of Section 2 for integro-sum functional inequalities with delay.