Available online at www.sciencedirect.com
ScienceDirect
Nonlinear Analysis 66 (2007) 2246–2254
Nonlinear Analysis
www.elsevier.com/locate/na

On existence and asymptotic behaviour of solutions of a functional integral equation

Józef Banaś[a,*], Ignacio J. Cabrera[b]

[a] Department of Mathematics, Rzeszów University of Technology, W. Pola 2, 35-959 Rzeszów, Poland
[b] Department of Mathematics, University of Las Palmas de Gran Canaria, Campus de Tafira Baja, 35017 Las Palmas de Gran Canaria, Spain

Received 31 October 2005; accepted 9 March 2006

Abstract

The paper contains a result on the existence and asymptotic behaviour of solutions of a functional integral equation. That result is proved under rather general hypotheses. The main tools used in our considerations are the concept of a measure of noncompactness and the classical Schauder fixed point principle. The investigations of the paper are placed in the space of continuous and tempered functions on the real half-line. We prove an existence result which generalizes several ones concerning functional integral equations and obtained earlier by other authors. The applicability of our result is illustrated by some examples.

Keywords: Functional integral equation; Asymptotic behaviour; Fixed point theorem; Measure of noncompactness

1. Introduction

The theory of functional integral equations creates an important branch of nonlinear analysis. Equations of such a type are often applicable in physics, mechanics, control theory, biology, economics and engineering, for instance (cf. [1–3,5,7–9,11,16]).

It is worthwhile mentioning that several important problems of the theory of ordinary and delay differential equations lead to investigations of functional integral equations of various types [1–3,9,10,12,13,15,17,18].

* Corresponding author.
E-mail address: jbanas@prz.rzeszow.pl (J. Banaś).

0362-546X/$ - see front matter
doi:10.1016/j.na.2006.03.015

In this paper we will examine the following functional integral equation

$$x(t) = f\left(t, \int_0^t x(s)\mathrm{d}s, \quad \int_0^t x(h(s, x(s)))\mathrm{d}s\right), \quad t \geq 0. \tag{1.1}$$

The functional integral equation of the above form contains a lot of special types of functional integral equations. For example, differential equations with transformed argument or differential equations of neutral type can be transformed to equations of the type (1.1).

The goal of this paper is to investigate the problem of the existence of solutions of Eq. (1.1). Using the classical Schauder fixed point principle and the concept of a measure of noncompactness we show that Eq. (1.1) has solutions under rather general and convenient assumptions. We obtain also some asymptotic characterization of solutions of Eq. (1.1).

Let us note that our result concerning Eq. (1.1) admits several realizations which will be illustrated with the help of a few examples.

The functional integral equation of the type (1.1) or its special cases were investigated in a lot of papers and monographs (cf. [1–4,9–12,14,18]. The result obtained in this paper generalizes those obtained in the above mentioned research papers.

2. A few auxiliary facts

In this section we gather some facts which will be needed in our further considerations. Assume that E is a real Banach space with the norm $\|\cdot\|$ and the zero element θ. Denote by $B(x, r)$ the closed ball centered at x and with radius r. The ball $B(\theta, r)$ will be denoted by B_r. If X is a subset of E then the symbols $\overline{X}$ and $\operatorname{Conv} X$ stand for the closure and convex closure of X, respectively. The family of all nonempty and bounded subsets of E will be denoted by $\mathfrak{M}_E$ while its subfamily consisting of all relatively compact sets is denoted by $\mathfrak{N}_E$.

Following [6] we accept the following definition of a measure of noncompactness.

Definition 2.1. A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}_+ = [0, \infty)$ is said to be a measure of noncompactness if it satisfies the following conditions:

1^o The family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{N}_E$.
2^o $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.
3^o $\mu(\overline{X}) = \mu(\operatorname{Conv} X) = \mu(X)$.
4^o $\mu(\lambda X + (1 - \lambda)Y) \leq \lambda\mu(X) + (1 - \lambda)\mu(Y)$ for $\lambda \in [0, 1]$.
5^o If (X_n) is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ $(n = 1, 2, \ldots)$ and if $\lim_{n\to\infty} \mu(X_n) = 0$ then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

The family $\ker \mu$ defined in 1^o is called the kernel of the measure of noncompactness μ.

For further facts concerning measures of noncompactness we refer the reader to [6].

Now, let us assume that $p = p(t)$ is a given function defined and continuous on the interval $\mathbb{R}_+$ with real positive values. Denote by $C(\mathbb{R}_+, p(t)) = C_p$ the Banach space consisting of all real functions $x = x(t)$ defined and continuous on $\mathbb{R}_+$ and such that

$$\sup\{|x(t)|p(t) : t \geq 0\} < \infty.$$

The space C_p is furnished with the following standard norm

$$\|x\| = \sup\{|x(t)|p(t) : t \geq 0\}$$

(cf. [6] for details).