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A quasilinearization method for elliptic problems with a nonlinear boundary condition

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Abstract

We study a nonlinear elliptic second order problem with a nonlinear boundary condition. Assuming the existence of an ordered couple of a supersolution and a subsolution, we develop a quasilinearization method in order to construct an iterative scheme that converges to a solution. Furthermore, under an extra assumption we prove that the convergence is quadratic.

Keywords: Elliptic PDEs; Nonlinear boundary conditions; Quasilinearization method; Upper and lower solutions

1. Introduction

In this work, we study the following nonlinear elliptic boundary problem:

$$\begin{cases} \Delta u = f(x,u) & \text{in } \Omega \\ \dfrac{\partial u}{\partial \eta} = g(x,u) & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here Ω is a bounded smooth domain of $\mathbb{R}^n$, and $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are continuous and twice continuously differentiable with respect to u.

Nonlinear boundary conditions of this kind appear for example when one considers the problem of finding extremals for the best constant in the Sobolev trace inequality (see e.g. [5]

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and [11]). On the other hand, for $n = 1$, the problem can be regarded as a mathematical model for the axial deformation of a nonlinear elastic beam, with two nonlinear elastic springs acting at the extremities according to the law $u'(0) = -g(u(0))$, $u'(T) = g(u(T))$, and the total force exerted by the nonlinear spring undergoing the displacement u given by $f(t, u)$ [6,14].

The aim of this paper is to develop a quasilinearization technique for problem (1.1) assuming the existence of an ordered couple of a subsolution and a supersolution. More precisely, we construct an iterative scheme that converges to a solution. Furthermore, under an extra assumption we prove that the convergence is quadratic.

The method of supersolutions and subsolutions (definitions will be given in Section 2 below) is one of the most extensively used tools in nonlinear analysis, both for ODE and PDE problems. There exists a vast literature on this subject; see e.g. [4] for a survey. In particular, for elliptic problems with nonlinear boundary conditions such as (1.1), this method has been applied to obtain existence results for example in [7,12].

The quasilinearization method has been developed by Bellman and Kalaba [3], and generalized by Lakshmikantham [9,10]. It has been applied to different nonlinear problems in the presence of an ordered couple of a subsolution and a supersolution. In a recent work [8] it has been successfully applied for a second order ODE Neumann problem for the case in which the supersolution β and the subsolution α present the reversed order, namely $\beta \leq \alpha$.

Our main results read as follows.

Theorem 1.1. *Let $\alpha, \beta \in H^1(\Omega) \cap C(\overline{\Omega})$ be respectively a subsolution and a supersolution of (1.1) such that $\alpha \leq \beta$. Furthermore, assume that*

$$\frac{\partial^2 f}{\partial u^2}(x, u) \leq 0$$

for $x \in \overline{\Omega}$ and $\alpha(x) \leq u \leq \beta(x)$, and

$$\frac{\partial^2 g}{\partial u^2}(x, u) \geq 0$$

for $x \in \partial\Omega$ and $\alpha(x) \leq u \leq \beta(x)$. Then the sequence defined below by (3.1) and (3.2) converges in $H^1(\Omega) \cap C(\overline{\Omega})$ to a solution of (1.1).

The proof relies on an associated maximum principle and the unique solvability of the associated linear Robin problem.

Moreover, under a monotonicity condition on f and g, we prove the quadratic convergence of the method.

Theorem 1.2. *Under the hypotheses of the previous theorem, assume furthermore that*

$$\frac{\partial f}{\partial u}(x, u) > 0$$

for $x \in \overline{\Omega}$ and $\alpha(x) \leq u \leq \beta(x)$, and

$$\frac{\partial g}{\partial u}(x, u) < 0$$

for $x \in \partial\Omega$ and $\alpha(x) \leq u \leq \beta(x)$. Then the convergence of the sequence defined by (3.1) and (3.2) is quadratic for the $C(\overline{\Omega})$-norm.