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On the existence and uniqueness for higher order periodic boundary value problems☆

Yongxiang Li

Department of Mathematics, Northwest Normal University, Lanzhou 730070, People's Republic of China

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Abstract

This paper discusses the existence and uniqueness for the nth-order periodic boundary value problem

$$L_n u(t) = f(t, u(t)), \quad 0 \le t \le 2\pi,$$
$$u^{(i)}(0) = u^{(i)}(2\pi), \quad i = 0, 1, \ldots, n-1,$$

where $L_n u(t) = u^{(n)}(t) + \sum_{i=0}^{n-1} a_i u^{(i)}(t)$ is an nth-order linear differential operator, $n \ge 2$, and $f : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}$ is continuous. In the case that L_n has an even order derivative, we present some new spectral conditions for the nonlinearity $f(t, u)$ to guarantee the existence and uniqueness. These spectral conditions allow $f(t, u)$ to be a superlinear growth, and are the extension for the spectral separation condition presented recently in [Y. Li, Existence and uniqueness for higher order periodic boundary value problems under spectral separation conditions, J. Math. Anal. Appl. 322 (2) (2006) 530–539].

Keywords: nth-order PBVP; Existence and uniqueness; Spectral condition; Normal operator; Fixed-point theorem

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1. Introduction and main results

In this paper, we develop the existence and uniqueness results in [11] for the nth-order periodic boundary value problem (PBVP)

$$\begin{cases} L_n u(t) = f(t, u(t)), & 0 \le t \le 2\pi, \\ u^{(i)}(0) = u^{(i)}(2\pi), & i = 0, 1, \ldots, n-1, \end{cases} \tag{1}$$

where $L_n u(t) = u^{(n)}(t) + \sum_{i=0}^{n-1} a_i u^{(i)}(t)$ is an nth-order linear differential operator, $a_i \in \mathbb{R}$, $i = 0, 1 \ldots, n-1$, $n \ge 2$, $f : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}$ is continuous.

The special case of PBVP (1) for $n = 2$, the second-order periodic boundary value problem

$$\begin{cases} u''(t) = f(t, u(t)), & 0 \le t \le 2\pi, \\ u(0) = u(2\pi), & u'(0) = u'(2\pi) \end{cases} \tag{2}$$

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E-mail address: liyx@nwnu.edu.cn.

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has been widely investigated by many authors. One of the well-known results is that if f satisfies the nonresonance condition

$$-(N+1)^2+\varepsilon \le f_u(t,u) \le -N^2-\varepsilon,$$

where N is a nonnegative integer and ε is a positive constant, the PBVP (2) has a unique solution, see [1].

In [2], Cong extended this result to the periodic boundary value problem of the even order differential equation

$$\begin{cases} u^{(2n)}(t)+\sum_{j=1}^{n-1} c_j u^{(2j)}(t) = f(t,u(t)), \quad t\in[0,2\pi], \\ u^{(i)}(0)=u^{(i)}(2\pi), \quad i=0,1,\ldots,2n-1, \end{cases} \tag{3}$$

(to be rewritten) and showed that if the condition

$$\begin{aligned} 0 &\le N^{2n}+\sum_{j=1}^{n-1}(-1)^{j-n}c_j N^{2j}+\varepsilon \\ &\le (-1)^n f_u(t,u) \\ &\le (N+1)^{2n}+\sum_{j=1}^{n-1}(-1)^{j-n}c_j (N+1)^{2j}-\varepsilon \end{aligned}$$

holds for a nonnegative integer N and a positive constant ε, the PBVP (3) has a unique solution. Recently, this result has been extended to a more general even order differential equation by Chen and O'Regan in [15].

In [3], Cong, Huang and Shi considered the periodic boundary value problem of the odd order differential equation

$$\begin{cases} u^{(2n+1)}(t)+\sum_{j=0}^{n-1} c_j u^{(2j+1)}(t) = f(t,u(t)), \quad t\in[0,2\pi], \\ u^{(i)}(0)=u^{(i)}(2\pi), \quad i=0,1,\ldots,2n. \end{cases} \tag{4}$$

Under the condition that

$$m \le |f_u(t,u)| \le M, \qquad \forall (t,u)\in[0,2\pi]\times\mathbb{R},$$

(where m and M are positive constants) and other assumptions they obtained the existence and uniqueness results for the PBVP (4).

In a recent paper [11], the present author extends and improves the main results in [2] and [3] mentioned above. The new results concern the spectrum $\sigma(L_n)$ of L_n in periodic boundary condition, which is given by

$$\sigma(L_n)=\{P_n(k\mathbf{i})\,|\,k=0,\pm1,\pm2,\ldots\}, \tag{5}$$

where

$$P_n(\lambda)=\lambda^n+a_{n-1}\lambda^{n-1}+\cdots+a_1\lambda+a_0 \tag{6}$$

is the eigenpolynomial of differential operator L_n and $\mathbf{i}$ is the imaginary unit.

For $\mu\in\mathbb{C}$ and $R>0$, we use $\overline{B}(\mu,R)$ to denote the closed circle in complex plane $\mathbb{C}$ with centre μ and radius R. We introduce the notation of the difference quotient set

$$\Delta(f) := \left\{ \frac{f(t,u)-f(t,v)}{u-v} \,\middle|\, t\in I, u,v\in\mathbb{R}, u\ne v \right\}, \tag{7}$$

here $I=[0,2\pi]$. In [11] the following existence and uniqueness theorem is obtained:

Theorem A. *Let $f: I\times\mathbb{R}\to\mathbb{R}$ be continuous. If there exist $\mu\in\mathbb{R}$ and $R>0$ such that*

$$\Delta(f)\subset\overline{B}(\mu,R), \qquad \overline{B}(\mu,R)\cap\sigma(L_n)=\emptyset, \tag{8}$$

then the PBVP (1) has a unique solution.