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Viscosity approximation methods for countable families of nonexpansive mappings in Banach spaces

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Abstract

Let C be a nonempty closed convex subset of a Banach space E and let $\{S_n\}$ be a family of nonexpansive mappings of C into itself such that the set of common fixed points of $\{S_n\}$ is nonempty. We first introduce a sequence $\{x_n\}$ of C defined by $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S_n x_n \quad \text{for all } n \in \mathbb{N},$$

where $\{\alpha_n\} \subset (0, 1)$ and f is a contraction of C into itself. Further, we give the conditions of $\{\alpha_n\}$ and $\{S_n\}$ under which $\{x_n\}$ converges strongly to a common fixed point of $\{S_n\}$. This result generalizes the strong convergence theorem for nonexpansive mappings by Suzuki [T. Suzuki, A sufficient and necessary condition for Halpern-type strong convergence to fixed points of nonexpansive mappings, Proc. Amer. Math. Soc. 135 (2007) 99–106] and the strong convergence theorem for accretive operators by Kamimura and Takahashi [S. Kamimura, W. Takahashi, Weak and strong convergence of solutions to accretive operator inclusions and applications, Set-Valued Anal. 8 (2000) 361–374], simultaneously. Using this result, we improve and extend the two above-mentioned results.

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1. Introduction

Throughout this paper, let E be a real Banach space with norm $\|\cdot\|$ and let $\mathbb{N}$ be the set of all positive integers. Let C be a nonempty closed convex subset of E. Then, a mapping $T : C \to C$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for all } x, y \in C.$$

We denote by $F(T)$ the set of fixed points of T. On the other hand, an operator $A \subset E \times E$ is called accretive if for $(x_1, y_1), (x_2, y_2) \in A$, there exists $j \in J(x_1 - x_2)$ such that $(y_1 - y_2, j) \geq 0$, where J is the duality mapping on E. For an accretive operator $A \subset E \times E$ and $r > 0$, we can define a mapping $J_r : R(I + rA) \to D(A)$ by $J_r = (I + rA)^{-1}$,

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where $R(I + rA)$ and $D(A)$ are the range of $I + rA$ and the domain of A, respectively. An accretive operator A is said to be m-accretive if $R(I + rA) = E$ for all $r > 0$. Recently, Suzuki [32] proved the following strong convergence theorem for nonexpansive mappings in a Banach space; see also [38,29,35,40].

Theorem 1.1. *Let E be a reflexive Banach space with a uniformly Gatêaux differentiable norm. Let C be a nonempty closed convex subset of E which has the fixed-point property for nonexpansive mappings and let $T : C \to C$ be a nonexpansive mapping such that $F(T)$ is nonempty. Define a sequence $\{x_n\}$ of C as follows: $x_1, u \in C$ and*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)((1 - \lambda)x_n + \lambda T x_n) \quad \text{for all } n \in \mathbb{N},$$

where $\lambda \in (0, 1)$ and $\{\alpha_n\} \subset (0, 1)$ satisfies the following conditions:

$$\alpha_n \to 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Kamimura and Takahashi [9] also proved the following strong convergence theorem for accretive operators in a Banach space; see also [2,3,8,14,19,22,27,30].

Theorem 1.2. *Let E be a reflexive Banach space with a uniformly Gatêaux differentiable norm which has the fixed-point property for nonexpansive mappings. Let $A \subset E \times E$ be an m-accretive operator with $A^{-1}0 \neq \emptyset$. Define a sequence $\{x_n\}$ of E as follows: $x_1, u \in E$ and*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) J_{t_n} x_n \quad \text{for all } n \in \mathbb{N},$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{t_n\} \subset (0, \infty)$ satisfy the following conditions:

$$\alpha_n \to 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad t_n \to \infty.$$

Then, the sequence $\{x_n\}$ converges strongly to $u \in A^{-1}0$.

In this paper, motivated by Suzuki [32], Kamimura and Takahashi [9], Moudafi [15] and Xu [39], we prove a strong convergence theorem for countable families of nonexpansive mappings in a Banach space which unifies the results of [32,9]. Using this result, we improve and extend the results of [32,9]. The proof is closely related to Takahashi [36], Nakajo, Shimoji and Takahashi [18], and Kikkawa and Takahashi [10,11].

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* denote the dual of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists. In the case, E is called smooth. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in U$, the limit (2.1) is attained uniformly for $x \in U$. We know that if E is smooth, then the duality mapping J is single valued. Further, if the norm of E is uniformly Gâteaux differentiable, then J is uniformly norm to weak* continuous on each bounded subset of E; see [25,33]. Let C be a closed convex subset of E. A mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of all fixed points of T. Let I denote the identity operator on E. An operator $A \subset E \times E$ with domain $D(A) = \{x \in E : Az \neq \emptyset\}$ and