Available online at www.sciencedirect.com
ScienceDirect
Nonlinear Analysis 70 (2009) 936–951
Nonlinear Analysis
www.elsevier.com/locate/na

The Fermat rule for multifunctions for super efficiency

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Received 6 August 2007; accepted 15 January 2008

Abstract

Using variational analysis, we study the vector optimization problems with objectives being closed multifunctions on Banach spaces or in Asplund spaces. In terms of the coderivatives and normal cones, we present Fermat's rules as necessary or sufficient conditions for a super efficient solution of the above problems.

MSC: 49J52; 90C29

Keywords: Vector optimization; Multifunction; Super efficient solution; Coderivative

1. Introduction

Let X, Y be Banach spaces and $\Phi : X \to 2^Y$ be a closed multifunction. Let C be a nontrivial closed convex pointed $(C \cap (-C) = \{0\})$ cone in Y, which specifies a partial order $\leq_C$ on Y as follows: for $y_1, y_2 \in Y$,

$$y_1 \leq_C y_2 \quad \text{if and only if } y_2 - y_1 \in C.$$

Let Ω be a closed subset of X. The main objective of this paper is to study the following constrained vector optimization problem

$$C - \min_{x \in \Omega} \Phi(x). \tag{1.1}$$

In the special case when $\Omega = X$, (1.1) reduces to the following unconstrained vector optimization problem

$$C - \min_{x \in X} \Phi(x). \tag{1.2}$$

Let A be a subset of Y and $a \in A$. Recall that a is called a Pareto efficient point of A, written as $a \in E(A, C)$, if

$$(A - a) \cap (-C) = \{0\}.$$

Following Borwein and Zhuang [1], we say that a is a super efficient point of A if there exists a real number $L > 0$ such that

$$\mathrm{cl}[\mathrm{cone}(A - a)] \cap (B_Y - C) \subset L B_Y,$$

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doi:10.1016/j.na.2008.01.023

where B_Y denotes the closed unit ball of Y. We use $SE(A, C)$ to denote the set of all super efficient points of A. It is known and easy to verify that $a \in SE(A, C)$ if and only if there exists a constant $L > 0$ such that

$$\|x - a\| \le L\|y\| \quad \text{for all } x \in A \text{ and } y \in Y \text{ with } x - a \le_C y.$$

It follows that $SE(A, C) \subset E(A, C)$. The super efficiency refines the notion of efficiency and other kinds of proper efficiency and reflect many desirable properties to vector optimization problems. Many authors have studied the super efficiency [2–6]. For $\bar{x} \in \Omega$ and $\bar{y} \in \Phi(\bar{x})$, we say that $(\bar{x}, \bar{y})$ is a local super efficient solution of the multiobjective optimization problem (1.1) if there exists a neighborhood U of $\bar{x}$ such that

$$\bar{y} \in SE(\Phi(U \cap \Omega), C).$$

It is a surprise for us that there are only a few authors who address the issue of necessary conditions (for $(\bar{x}, \bar{y})$ to provide an efficient solution). Recently, Minami [7] studied multiobjective program on Banach space with a single-valued objective function and with finitely many equality/inequality constraints given by numerical functions. Under the convexity assumptions, Götz and Jahn [8] studied necessary optimality conditions for weak Pareto solution using the notion of contingent derivative. Very recently, Ye and Zhu [9] gave some necessary optimality conditions for single-valued vector optimization problems with respect to an abstract order in an Euclidean space setting. Single-valued vector optimization problems with respect to abstract order (regardless to linear structure) have also been discussed in Zhu [10] and Mordukhovich,Traiman and Zhu [11].

In the special case when $Y = R$, $C = [0, +\infty)$ and Φ is given by

$$\Phi(x) = [f(x), +\infty) \quad \text{for all } x \in X, \tag{1.3}$$

where $f : X \to R \cup \{+\infty\}$ is a proper lower semicontinuous function, it is easy to verify that $(\bar{x}, f(\bar{x}))$ is a local super (or Pareto) efficient solution of (1.2) if and only if $\bar{x}$ is a local minimum point of f. The following result is well known as the Fermat rule:

$$f \text{ attains a local minimum at } \bar{x} \Rightarrow 0 \in \partial_c f(\bar{x}),$$

where $\partial_c f(\bar{x})$ denotes the Clarke subdifferential of f at $\bar{x}$. Noting (see [12]) that $\partial_c f(\bar{x})$ and the associated coderivative $D_c^* \Phi(\bar{x}, f(\bar{x})) : R \to 2^{X^*}$ are related by

$$\partial_c f(\bar{x}) = D_c^* \Phi(\bar{x}, f(\bar{x}))(1),$$

Zheng and Ng [13] established the Fermat rule for multifunctions for Pareto efficiency. Under some restricted conditions (eg. the ordering cone has a nonempty interior), Zheng and Ng [13] showed that if $(\bar{x}, \bar{y}) \in \mathrm{Gr}(\Phi)$ is a local Pareto efficient solution of (1.2), then there exists $c^* \in C^+$ with $\|c^*\| = 1$ such that

$$0 \in D_c^* \Phi(\bar{x}, \bar{y})(c^*),$$

where $C^+ := \{y^* \in Y^* : \langle c^*, c\rangle \ge 0 \text{ for all } c \in C\}$. In this paper, we establish the corresponding Fermat rule for super efficiency for the vector optimization problem (1.1) and (1.2) by the coderivatives in Clarke's sense and in Mordukhovich's sense, respectively. In Section 3, we present fuzzy Fermat's rules for the constrained problem (1.1). In Section 4, under some restricted conditions, we present exact Fermat's rules for the problem (1.1). In Section 5, we give an sufficient condition for local super efficient solutions for the unconstrained problem (1.2).

2. Preliminaries

In this section, we assume that X is a Banach space. Let $f : X \to R \cup \{+\infty\}$ be a proper lower semicontinuous function. Let $x \in \mathrm{dom}(f) := \{x \in X : f(x) < +\infty\}$, let $h \in X$, and let $f^0(x, h)$ denote the generalized directional derivative given by Rockafellar [12], that is,

$$f^0(x, h) := \lim_{\varepsilon \downarrow 0} \limsup_{z \xrightarrow{f} x, t \downarrow 0} \inf_{w \in h + \varepsilon B_X} \frac{f(z + tw) - f(z)}{t},$$