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Existence and regularity for a nonlinear boundary flow problem of population genetics

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Abstract

In this paper we consider a heat equation with nonlinear boundary condition occurring in population genetics, the selection–migration problem for alleles in a region, considering flow of genes throughout the boundary. Such a problem determines a gradient flow in a convenient phase space and then the dynamics for large times depends heavily on the knowledge of the equilibrium solutions. We address the questions of the existence of a nontrivial equilibrium solution and its regularity.

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1. Introduction

A model in population genetics which describes the changes of gene frequency in a population living in a region of space, considering the effects of gene flows and natural selection within that region, was introduced by Fleming [14] in a more general manner than that of Fisher's proposal in [13], and studied by several authors, for instance, [7–9,15,16,18] and references therein. In this paper we consider a model analogous to that of [14] considering flow of genes throughout the boundary of the region, given by

$$\begin{cases} \partial_t u = \Delta u & \text{in } \Omega \times \mathbb{R}^+ \\ \dfrac{\partial u}{\partial \nu} = \lambda s(x) f(u) & \text{on } \partial\Omega \times \mathbb{R}^+, \end{cases} \tag{1.1}$$

with the initial datum taken in a suitable phase space. Here $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$ and ν is the outward normal to $\partial\Omega$. In the model, $u(x,t)$ is the gene frequency at time t and position $x \in \overline{\Omega}$, $\lambda > 0$ is a parameter; the nonlinear reaction at the boundary represents the effects of natural selection and is such that the C^2-function $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies

$$\text{(H-1)} \quad \begin{cases} \bullet\ f > 0 & \text{in } (0,1), \\ \bullet\ f(0) = 0 = f(1), \\ \bullet\ f'(0) > 0,\ f'(1) < 0, \end{cases}$$

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and the boundary weight function $s : \partial\Omega \longrightarrow \mathbb{R}$ verifies

$$\text{(H-2)} \quad \begin{cases} \bullet\ s(\cdot) \in C^{1,\theta}(\partial\Omega), & \text{for } 0 < \theta < 1, \\ \bullet\ s(\cdot) & \text{changes sign in } \partial\Omega, \\ \bullet\ \displaystyle\int_{\partial\Omega} s(x)\, \mathrm{d}\mathcal{H}^{n-1} < 0, \end{cases}$$

where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure. The equilibrium solutions of (1.1) are the solutions of the elliptic problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = \lambda s(x) f(u) & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

that lie in the appropriate phase space described in Section 2.

The only constant solutions of (1.2) which take values between 0 and 1 are $u \equiv 0$ and $u \equiv 1$, the zeros of f in [0, 1], called the trivial equilibrium solutions of (1.1). In this work we address the questions of existence and regularity of nontrivial equilibrium solutions of (1.1) that are physically meaningful solutions of the elliptic problem (1.2), that is, solutions of (1.2) which take values on the interval [0, 1] because u is a gene frequency in the model. Such questions have great importance because, according to Section 2, problem (1.1) generates a nonlinear dynamical system in a suitable phase space, which is a gradient system; see also [3,5,10,15]. Therefore, the knowledge of the equilibrium solutions is of vital importance in studying the dynamics of (1.1) since all its solutions approach the set of equilibrium solutions when t is large.

Problem (1.1) is also studied in [17] where questions like those of bifurcation of equilibria and their stability are investigated. Moreover, it is part of a topic of research interest in differential equations, namely, parabolic equations with nonlinear boundary conditions. There are very many works on that subject, but approaches of great generality can be found in [3,5,6,10] and in references therein.

The remainder of the paper is organized in the following way. In Section 2, we consider the nonlinear dynamical system generated by (1.1) in a closed and positively invariant subset of $H^1(\Omega)$ via the maximum principle. The phase space accords with the physical characteristics of the model that demands initial data and evolution taking values on the interval [0, 1]. In Section 3, we prove the existence of a nontrivial equilibrium solution of (1.1) in the case of the parameter $\lambda > 0$ being not small. Such a solution is obtained by direct minimization of the energy functional corresponding to (1.2) and is proved to be nonconstant using an adequate admissible function to draw inferences concerning its energy level. When $\lambda > 0$ is small enough, the authors of [17] prove, with the help of the Implicit Function Theorem, that the only equilibrium solutions of (1.1) are the constant ones. In Section 4, we prove directly that weak solutions of the elliptic problem (1.2), not necessarily being equilibrium solutions of (1.1) when we assume, for example,

$$\text{(H-3)} \quad f, f' \in L^\infty(\mathbb{R}),$$

are actually classical solutions, specifically, in the class $C^{2,\theta}(\overline{\Omega})$ for some $0 < \theta < 1$. In the case of a weak equilibrium solution of (1.1), hypothesis (H-3) is not necessary. The proof relies on a bootstrap argument connected to a regularity result from the L^p-theory of nonhomogeneous elliptic boundary value problems found in [12]. That way of using the bootstrap for elliptic problems under nonhomogeneous, in particular nonlinear, boundary conditions seems to be applicable to problems whose nonlinearities satisfy more general growth conditions.

2. The nonlinear dynamical system

In this section we see how problem (1.1) can be formulated in a weak sense and written like an abstract evolution equation, and consider a suitable phase space in which (1.1) generates a nonlinear dynamical system through a closed and positively invariant subset of $H^1(\Omega)$.

Following the approaches in [5,10], (1.1) can be viewed as an abstract evolution equation of the form

$$\begin{cases} \partial_t u + \mathcal{A}u = \mathcal{F}(u) \\ u(0) = u_0 \in H^1(\Omega) \end{cases}$$