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Boundedness and blowup for nonlinear degenerate parabolic equations

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Abstract

The author deals with the quasilinear parabolic equation $u_t = [u^\alpha + g(u)]\Delta u + bu^{\alpha+1} + f(u, \nabla u)$ with Dirichlet boundary conditions in a bounded domain Ω, where f and g are lower-order terms. He shows that, under suitable conditions on f and g, whether the solution is bounded or blows up in a finite time depends only on the first eigenvalue of $-\Delta$ in Ω with Dirichlet boundary condition. For some special cases, the result is sharp.

Keywords: Porous medium equation; Quasilinear parabolic equation; Global existence; Blowup solutions

In this paper we consider the following quasilinear parabolic equation:

$$u_t = [u^\alpha + g(u)]\Delta u + bu^{\alpha+1} + f(u, \nabla u), \quad t > 0, x \in \Omega, \tag{1}$$

with the initial and boundary conditions

$$\begin{cases} u(x,t) = 0, & t > 0, x \in \partial\Omega, \\ u(x,0) = \phi(x) > 0, & x \in \Omega, \end{cases} \tag{2}$$

where $\Omega \subset R^n$ is a bounded domain with smooth boundary $\partial\Omega$.

The study of (1) and (2) is of great interest due to its applications. If $g(0) = 0$, (1) is the well-known porous medium equation with a source and a convection term (see [1,5,7,8,18,20,22]). In the simple cases where $g = f = 0$, many authors (see [2,15,22]) proved that whether the solution exists globally or blows up in a finite time depends only on the first eigenvalue of $-\Delta$ in the domain Ω with Dirichlet boundary condition.

Ding et al. [11,12] discussed the general quasilinear parabolic equations and obtained some interesting blowup results. Suzuki [23] considered the following equation

$$u_t = \Delta u^m + a \cdot \nabla u^q + u^p,$$

where $1 \le m < p < m + 2/(N+1)$ and $(m+1)/2 \le q < (m+p)/2$. He obtained a universal bound which is independent of the initial data for all global nonnegative solutions of the Dirichlet problem. Chen and Yu [5] dealt

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with the following equation

$$u_t = \nabla \cdot (g(u)\nabla u) + h(u, \nabla u) + f(u)$$

with $u|_{\partial\Omega} = 0, u(x, 0) = \phi(x)$. If f, g and h are polynomials with proper degrees and proper coefficients, they showed that the blowup property depends only on the first eigenvalue of $-\Delta$ in Ω with Dirichlet boundary condition. For a special case, they obtain a sharp result. Many authors also discussed the blowup properties of local or nonlocal sources or degenerate parabolic systems and obtained similar interesting results (see [6,9,10,13,14,16,19,21]).

In this paper, we use a new method to obtain both lower and upper bound for the solution of (1). In this method, we estimate the integral of a ratio of one solution to the other. This method shows successful in proving existence and blowup problems (see [3,5]), especially, the problems where the comparison principle is failed (see [4]). We first prove the boundedness of the solutions and then the blowup properties under different conditions on f and g for the problem (1) and (2). In some special cases, the result is sharp.

Throughout this paper, we denote by $\lambda_1 = \lambda_1(\Omega)$ the first eigenvalue of

$$\Delta\psi + \lambda_1\psi = 0 \quad \text{with } \psi|_{\partial\Omega} = 0, \tag{3}$$

where ψ is the first normalized eigenfunction.

Theorem 1. *Suppose that, for any ξ, $\varepsilon > 0$ and $u > 0$, there exist positive constants $c = c(\varepsilon)$ and c_0 such that*

$$|f(u, \xi)| \le \varepsilon u^{\alpha-1}(u^2 + |\xi|^2) + c(\varepsilon)u^{-\sigma}, \tag{4}$$

and

$$g(u) \ge c_0 u^{\beta} \quad \text{and} \quad g(u) \ge c_0 u g'(u) \ge 0, \tag{5}$$

where $\beta \ge 0$ and $\sigma \ge 0$. If $b < \lambda_1$, then, for any initial value $\phi(x) \in C_0^{1+\eta}(\overline{\Omega})$ with $\eta > 0$, the solution of the problem (1) and (2) is bounded for all $t > 0$.

Proof. By Lemma 1 in [5], the solution u is positive in Ω. This guarantees that u is a classical solution as long as u is bounded. By the continuity of the eigenvalues with respect to the domain Ω (see [17]), we can find a small $\delta > 0$ such that $b < \lambda_1 - \delta$ and we can define a domain $D \supset \Omega$ such that $\partial\Omega$ is located inside of D and there exists the first normalized eigenfunction of

$$\Delta\psi_1 + (\lambda_1 - \delta)\psi_1 = 0 \tag{6}$$

in D with $\psi_1|_{\partial D} = 0$. Then ψ_1 is positive in $\overline{\Omega}$. For any positive number n, define

$$w_n(t) = \int_\Omega \frac{u^n(x, t)}{\psi_1^{kn}(x)} \mathrm{d}x, \tag{7}$$

where $k < 1$ satisfying $b < \lambda_1 k - \delta$. Differentiating (7), substituting the equation of (1) into the result integral, integrating by parts and using (4), we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} w_n(t) = {} & n \int_\Omega \frac{u^{n-1}}{\psi_1^{kn}} \{[u^\alpha + g(u)]\Delta u + bu^{\alpha+1} + f(u, \nabla u)\}\mathrm{d}x \\
\le {} & -n(n-1+\alpha) \int_\Omega \frac{u^{n-2+\alpha}}{\psi_1^{kn}} |\nabla u|^2 \mathrm{d}x + kn^2 \int_\Omega \frac{u^{n-1+\alpha}}{\psi_1^{kn+1}} \nabla u \nabla \psi_1 \mathrm{d}x \\
& - n(n-1) \int_\Omega \frac{u^{n-2}}{\psi_1^{kn}} g(u)|\nabla u|^2 \mathrm{d}x + kn^2 \int_\Omega \frac{u^{n-1}}{\psi_1^{kn+1}} g(u)\nabla u \nabla \psi_1 \mathrm{d}x \\
& - n \int_\Omega \frac{u^{n-1}}{\psi_1^{kn}} g'(u)|\nabla u|^2 \mathrm{d}x + nb \int_\Omega \frac{u^{n+\alpha}}{\psi_1^{kn}} \mathrm{d}x \\
& + n \int_\Omega \frac{u^{n-1}}{\psi_1^{kn}} [\varepsilon(u^{\alpha+1} + u^{\alpha-1}|\nabla u|^2) + c(\varepsilon)u^{-\sigma}]\mathrm{d}x.
\end{aligned} \tag{8}$$