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General comparison principle for quasilinear elliptic inclusions

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Abstract

The main goal of this paper is to prove existence and comparison results for elliptic differential inclusions governed by a quasilinear elliptic operator and a multivalued function given by Clarke's generalized gradient of some locally Lipschitz function. These kinds of problems have been treated in the past by various authors including the authors of this paper. However, in all the works we are aware of, additional assumptions on the structure of the elliptic operator and/or the generalized Clarke's gradient are needed to get comparison results in terms of sub-supersolutions. Comparison principles were obtained recently, e.g., in the case where the elliptic operator is of potential type, or Clarke's gradient is required to satisfy some one-sided growth condition, or the sub-supersolutions are supposed to satisfy additional properties. The novelty of this paper is that we are able to obtain a comparison principle without assuming any of the above restrictions. To the best of our knowledge this is the first mathematical treatment of the considered elliptic inclusion in its full generality. The obtained results of this paper complement the development of the sub-supersolution method for nonsmooth problems presented in a recent monograph by S. Carl, Vy K. Le and D. Motreanu.

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$, and let $V = W^{1,p}(\Omega)$ and $V_0 = W_0^{1,p}(\Omega)$, $1 < p < \infty$, denote the usual Sobolev spaces with their dual spaces V^* and V_0^*, respectively. In this paper we deal with the following quasilinear elliptic inclusion under Dirichlet boundary conditions:

$$u \in V_0 : \quad Au + \partial j(\cdot, u) \ni 0 \quad \text{in } V_0^*, \tag{1.1}$$

where A is a second-order quasilinear elliptic differential operator of the form

$$Au(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, \nabla u(x)), \quad \text{with } \nabla u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N}\right),$$

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and the function $j : \Omega \times \mathbb{R} \to \mathbb{R}$ is assumed to be measurable in $x \in \Omega$ for all $s \in \mathbb{R}$, and locally Lipschitz in $s \in \mathbb{R}$ for almost all (a.a.) $x \in \Omega$. The multivalued function $s \mapsto \partial j(x, s)$ that appears in (1.1) is given by Clarke's generalized gradient of the locally Lipschitz function $s \mapsto j(x, s)$, which is defined by

$$\partial j(x, s) := \{\zeta \in \mathbb{R} : j^o(x, s; r) \geq \zeta r, \forall\, r \in \mathbb{R}\}$$

for a.a. $x \in \Omega$, with $j^o(x, s; r)$ denoting the generalized directional derivative of $s \mapsto j(x, s)$ at s in the direction r given by

$$j^o(x, s; r) = \limsup_{y \to s, t \downarrow 0} \frac{j(x, y + tr) - j(x, y)}{t},$$

(cf., e.g., [6, Chap. 2]). If j has the form of a primitive of some Carathéodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$, i.e.,

$$j(x, s) := \int_0^s f(x, t)\mathrm{d}t$$

then the function $s \mapsto j(x, s)$ is continuously differentiable. Thus $\partial j(x, s) = \{\partial j(x, s)/\partial s\} = \{f(x, s)\}$, and (1.1) reduces to the elliptic boundary value problem

$$u \in V_0 : \quad Au + f(\cdot, u) = 0 \quad \text{in } V_0^*. \tag{1.2}$$

The method of sub-supersolutions for (1.2) with a Carathéodory function f is well known. A detailed exposition even for much more general elliptic problems than (1.2) can be found, e.g., in [3, Chap. 3]. In case f is merely bounded on bounded sets, i.e., if $f \in L^\infty_{\mathrm{loc}}(\Omega \times \mathbb{R})$ then $s \mapsto j(x, s)$ is locally Lipschitz and the generalized gradient $s \mapsto \partial j(x, s)$ can be characterized as follows: Define for every $(x, t) \in \Omega \times \mathbb{R}$,

$$f_1(x, t) := \lim_{\delta \to 0^+} \operatorname*{essinf}_{|\tau - t| < \delta} f(x, \tau), \qquad f_2(x, t) := \lim_{\delta \to 0^+} \operatorname*{ess\,sup}_{|\tau - t| < \delta} f(x, \tau),$$

then Proposition 1.7 in [9] ensures that

$$\partial j(x, s) = [f_1(x, s), f_2(x, s)]. \tag{1.3}$$

For $A = -\Delta_p$ and Clarke's gradient given in the form (1.3) a comparison result in terms of appropriately defined sub-supersolutions was recently obtained in [1]. The approach in [1] requires additional assumptions. First, the operator $A = -\Delta_p$ is a potential operator which allows us to apply variational methods, in particular, the interpretation of a solution u of (1.1) as a critical point of an associated (nonsmooth) energy functional. Second, in [1] the comparison result can only be obtained if a notion of sub and supersolution is used which is more restrictive than the notion given here, see Remark 2.2.

Comparison principles for general elliptic operators A, and Clarke's gradient $s \mapsto \partial j(x, s)$ satisfying a one-sided growth condition in the form

$$\eta_1 \leq \eta_2 + c(s_2 - s_1)^{p-1}, \tag{1.4}$$

for all $\eta_i \in \partial j(x, s_i)$, $i = 1, 2$, and for all s_1, s_2 with $s_1 < s_2$ can be found in [3]. Finally, assuming $s \mapsto \partial j(x, s)$ to be given in the form (1.3) and defining sub-supersolutions by elliptic inequalities related with the functions $f_i : \Omega \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, comparison principles for general operators A were obtained in [5, Chap. 6].

The main goal and the novelty of this paper are to establish a comparison principle for (1.1) in terms of sub-supersolution without assuming any of the above-mentioned restrictions on the structure of A and ∂j. It should be mentioned that the results obtained in this paper can be extended to more general elliptic operators A of the Leray–Lions type such as

$$Au(x) = -\sum_{i=1}^N \frac{\partial}{\partial x_i} a_i(x, u, \nabla u(x)) + a_0(x, u, \nabla u(x)),$$

and for nonlinear mixed boundary conditions as well. To the best of our knowledge this is the first rigorous mathematical treatment of the considered elliptic inclusion in its full generality as far as existence and comparison