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On maximal element theorems, variants of Ekeland's variational principle and their applications

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Abstract

In this paper, we establish several different versions of generalized Ekeland's variational principle and maximal element theorem for τ-functions in $\lesssim$ complete metric spaces. The equivalence relations between maximal element theorems, generalized Ekeland's variational principle, generalized Caristi's (common) fixed point theorems and nonconvex maximal element theorems for maps are also proved. Moreover, we obtain some applications to a nonconvex minimax theorem, nonconvex vectorial equilibrium theorems and convergence theorems in complete metric spaces.

MSC: primary 47H10; 58E30; 90C26; secondary 49J35; 54H25; 91B52

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1. Introduction

The famous Ekeland variational principle (in short EVP) [11–13] is a forceful tool in various fields of applied mathematical analysis and nonlinear analysis. It is well-known that the primitive EVP is equivalent to Caristi's fixed point theorem [1,5,7,17,18,22,25,29], to the drop theorem [15,23], to the petal theorem [15,23], and to Takahashi's nonconvex minimization theorem [16,18,24,29]; see e.g. [15]. A number of generalizations in various different directions of these results in metric (or quasi-metric) spaces and more generally in topological vector spaces have been investigated by several authors in the past; see [1–3,6–10,16–22,24–29] and references therein. In 1963, Bishop and Phelps [4] proved a fundamental theorem concerning the density of the set of support points of a closed convex subset of a Banach space by using an existence theorem of maximal elements in certain partially ordered complete subsets of a normed linear space. Granas and Horvath [14] studied a so-called Cantor space and obtained some maximal element theorems and fixed point theorems which can be applied to EVP and their equivalent formulations in complete metric spaces. Park [22] also gave generalized forms of EVP and some equivalents.

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The paper is divided into six sections. In Section 2, we first establish a τ-function version of the EVP and maximal element theorem (Theorem 2.1). In Section 3, we prove some equivalent formulations of Theorem 2.1. In Section 4, we give another type of τ-function version of the EVP and maximal element theorem. Certainly, some equivalent formulations of it can be established. In Section 5, we obtain a vectorial version of the EVP and maximal element theorem and vectorial versions of Caristi's common fixed point theorems. Finally, in Section 6, we establish nonconvex minimax theorems, nonconvex (vectorial) equilibrium theorems and convergence theorems in complete metric spaces. Consequently, in the paper, some of our results are original in the literature and we obtain many results in the literature as special cases; see e.g. [1–3,6–13,15–18,20–23,27–29] and references therein.

2. Preliminaries

Let X be a nonempty set and "$\lesssim$" a quasi-order (preorder or pseudo-order; that is, a reflexive and transitive relation) on X. Then $(X, \lesssim)$ is called a quasi-ordered set. Let (X, d) be a metric space with a quasi-order $\lesssim$. A nonempty subset M of X is said to be $\lesssim$ *complete* if every nondecreasing Cauchy sequence in M converges. An element v in X is called a *maximal element* of X if there is no element x of X, different from v, such that $v \lesssim x$; that is, $v \lesssim w$ for some $w \in X$ implies that $v = w$. An extended real valued function $f : X \to (-\infty, +\infty]$ is said to be

(i) *lower semicontinuous from above* (in short *lsca*) at $x_0 \in X$ [7,17] if for any sequence $\{x_n\}$ in X with $x_n \to x_0$ and $f(x_1) \geq f(x_2) \geq \cdots \geq f(x_n) \geq \cdots$ imply that $f(x_0) \leq \lim_{n\to\infty} f(x_n)$;
(ii) *upper semicontinuous from below* (in short *uscb*) at $x_0 \in X$ [17] if for any sequence $\{x_n\}$ in X with $x_n \to x_0$ and $f(x_1) \leq f(x_2) \leq \cdots \leq f(x_n) \leq \cdots$ imply that $f(x_0) \geq \lim_{n\to\infty} f(x_n)$.

The function f is said to be *lsca* (resp. *uscb*) on X if f is *lsca* (resp. *uscb*) at every point of X. It is obvious that the lower (resp. upper) semicontinuity implies the lower (resp. upper) semicontinuity from above (resp. below), but the reverse is not true (see [7, Example 1.3]). The function f is said to be proper if $f \not\equiv \infty$. Recall that a function $p : X \times X \to [0, \infty)$ is called a τ-*function* [17], if the following conditions hold:

(τ1) $p(x, z) \leq p(x, y) + p(y, z)$ for all $x, y, z \in X$;
(τ2) If $x \in X$ and $\{y_n\}$ in X with $\lim_{n\to\infty} y_n = y$ such that $p(x, y_n) \leq M$ for some $M = M(x) > 0$, then $p(x, y) \leq M$;
(τ3) For any sequence $\{x_n\}$ in X with $\lim_{n\to\infty} \sup\{p(x_n, x_m) : m > n\} = 0$, if there exists a sequence $\{y_n\}$ in X such that $\lim_{n\to\infty} p(x_n, y_n) = 0$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$;
(τ4) For $x, y, z \in X$, $p(x, y) = 0$ and $p(x, z) = 0$ imply $y = z$.

It is known that any w-distance, introduced and studied by Kada et al. [16], is a τ-function; see [17, Remark 2.1].
The following lemmas are crucial in our proofs. Below, unless otherwise specified, we shall assume that (X, d) is a metric space.

Lemma 2.1. *Let $p : X \times X \to [0, \infty)$ be a function. Then the following conclusions hold:*

(a) *Assume that p satisfies the condition (τ3). If a sequence $\{x_n\}$ in X with $\lim_{n\to\infty} \sup\{p(x_n, x_m) : m > n\} = 0$, then $\{x_n\}$ is a Cauchy sequence in X;*
(b) *If $p(x, \cdot)$ is lsc for each $x \in X$, then p satisfies the condition (τ2).*

Proof. The proof of (a) is similar to [17, Lemma 2.1]. The proof of (b) is straightfoward and is omitted. □

Lemma 2.2. *Suppose that the function $p : X \times X \to [0, \infty)$ satisfies the conditions (τ1) and (τ4) and the function $q : X \times X \to (-\infty, \infty]$ satisfies $q(x, x) \geq 0$ for all $x \in X$ and $q(x, z) \leq q(x, y) + q(y, z)$ for all $x, y, z \in X$. For each $x \in X$, let $G : X \to 2^X$ be defined by*

$$G(x) = \{y \in X : y \neq x,\ p(x, y) + q(x, y) \leq 0\}.$$

If $G(x)$ is nonempty for some $x \in X$, then for each $y \in G(x)$, we have $q(x, y) \leq 0$ and $G(y) \subseteq G(x)$.

Proof. Let $y \in G(x)$. Then $y \neq x$, $p(x, y) + q(x, y) \leq 0$ and $q(x, y) \leq 0$. If $G(y) = \emptyset$, then we are done. If $G(y) \neq \emptyset$, let $z \in G(y)$. Then $z \neq y$, $p(y, z) + q(y, z) \leq 0$ and $q(y, z) \leq 0$. It follows that

$$p(x, z) + q(x, z) \leq p(x, y) + p(y, z) + q(x, y) + q(y, z) \leq 0. \tag{2.1}$$