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Existence of a global attractor for a p-Laplacian equation in $\mathbb{R}^n$

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Abstract

In this paper, we prove the existence of the $(L^2(\mathbb{R}^n), W^{1,p}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n))$-global attractor for the p-Laplacian equation $u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u + f(u) = g$ with a more general nonlinear term $f = f_1 + a(x) f_2$ in $\mathbb{R}^n$, where $a \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and f_1, f_2 satisfy the arbitrary q-order polynomial growth condition without any restriction on q, p ($q \geq 2$, $p > 2$).

MSC: 35K57; 35B40; 35B41

Keywords: Global attractors; p-Laplacian equation; Asymptotic compactness

1. Introduction

In this paper, we prove the existence of global attractors in $W^{1,p}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ for the following p-Laplacian equation:

$$u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u + f(u) = g \qquad \text{in } \mathbb{R}^n \times \mathbb{R}^+, \tag{1.1}$$

with initial data condition

$$u(x, 0) = u_0(x), \tag{1.2}$$

where $p > 2, \lambda > 0$, $f = f_1 + a(x) f_2$ is a $\mathcal{C}^1$ function which satisfies the following assumptions,

$$f(0) = 0, \tag{1.3}$$

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$$\alpha_1|u|^q - \beta_1|u|^p \le f_1(u)u \le \gamma_1|u|^q + \delta_1|u|^p, \qquad q \ge 2 \text{ and } p > 2, \tag{1.4}$$
$$\lambda > \beta_1 \qquad \text{and} \qquad f_1'(u) \ge -c_1, \tag{1.5}$$
$$\alpha_2|u|^q - \beta_2 \le f_2(u)u \le \gamma_2|u|^q + \delta_2, \qquad q \ge 2, \tag{1.6}$$
$$f_2'(u) \ge -c_2, \tag{1.7}$$
$$a \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n), \qquad a(x) > 0 \qquad \text{for any } x \in \mathbb{R}^n, \tag{1.8}$$

where $\alpha_i, \beta_i, \gamma_i, \delta_i$ and c_i $(i = 1, 2)$ are positive constants.

Here $g \in L^s(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, s satisfies

$$\frac{1}{s} \in \begin{cases} \left[1 - \frac{1}{p}, 1 - \frac{1}{p} + \frac{1}{n}\right], & n > p \\ \left[1 - \frac{1}{p}, 1\right), & n = p \quad \text{or} \quad \frac{1}{s} + \frac{1}{q} = 1 \\ \left[1 - \frac{1}{p}, 1\right], & n < p. \end{cases} \tag{1.9}$$

It is interesting to note that the nonlinear term f is more general. For example, if $f_1 \equiv 0$, a is the characteristic function for a smooth bounded domain $\Omega \subset \mathbb{R}^n$, that is,

$$a(x) = \begin{cases} 1, & x \in \Omega, \\ 0, & x \notin \Omega, \end{cases} \tag{1.10}$$

then we see that the assumptions imposed on the nonlinear term f (i.e. f_2) in this paper are the same as in [1,3,19] for the case of bounded domains.

The existence of a global attractor for Eq. (1.1) in a bounded domain Ω has been studied extensively in many monographs and lectures. In [1], the authors give a more detailed discussion of this problem, however, the existence of an $(L^2(\Omega), W_0^{1,p}(\Omega) \cap L^q(\Omega))$-global attractor remains unknown for the case $2 < p < n$ and $q > \frac{np}{n-p}$; the authors in [6] obtain that the corresponding semigroup has an $(L^2(\Omega), W_0^{1,p}(\Omega) \cap L^q(\Omega))$-global attractor under some additional conditions, i.e., either assume that $p > \frac{n}{2}$ and that $f = h_1 + h_2$, where h_1 satisfies $(h_1, u) > 0$ and h_2 is a global $(L^2(\Omega), L^2(\Omega))$-Lipschitz mapping, or assume that f satisfies some growth condition such that it can be dominated by the p-Laplacian operator. For some other results concerning this problem in bounded domains, see [3–6,17,19] and the references therein.

It is well known that when we consider the asymptotic behavior of the solutions in an unbounded domain, particularly the existence of global attractors, we will face more difficulties due to the unboundedness of the spatial domain. The most difficult issue is the absence of the standard compact Sobolev embeddings of several functional spaces. In order to overcome the difficulty of the noncompact embedding, several methods have been developed. Many authors employ weighted spaces, locally uniform spaces and bounded continuous functional spaces, see, e.g., [2,8–10] and the references therein. However, when working in these spaces the initial data and forcing term are usually assumed to be in corresponding spaces. Recently, in [14,18] the authors used a suitable cut-off function to decompose the whole space $\mathbb{R}^n$ into a bounded ball and its complement, then the asymptotic compactness of the semigroup follows from the compact Sobolev embedding in the bounded ball and the estimates in its complement. Using this idea, for Eq. (1.1), the author in [12] obtained the existence of the $(L^2(\mathbb{R}^n), L^\infty(\mathbb{R}^n))$-global attractor when $n \le p$ and the existence of the $(L^2(\mathbb{R}^n), L^{\frac{np}{n-p}}(\mathbb{R}^n))$-global attractor when $n > p$. But he