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On the construction of boundary weak triangular norms through additive generators ☆

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Abstract

Recently, M. Urbański and J. Wąsowski [Fuzzy arithmetic based on boundary weak t-norms, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 13 (2005) 27–37] extended the triangular norm operators to a new class of operators called the boundary weak triangular norms. This extension is based on the replacement of the condition $T(x, 1) = x$ by the weaker one $T(x, 1) \leq x$ for $x \in [0, 1)$. In this paper we discuss the construction of the boundary weak triangular norms based on their additive generators. Continuous decreasing functions $f : [0, 1] \to [0, \infty]$ which generate proper boundary weak triangular norms are completely characterized. Several examples are included.

Keywords: Triangular norm; Boundary weak triangular norm; Additive generator; Pseudo-inverse

1. Introduction

A triangular norm (*t-norm* for short) $T : [0, 1]^2 \to [0, 1]$ is an operation on [0, 1] which is associative, commutative, non-decreasing and with neutral element 1, i.e., $([0, 1], T)$ is a fully ordered Abelian semigroup with neutral element 1 and annihilator 0. Triangular norms are applied in many fields such as probabilistic metric spaces, many-valued logics and their applications and the theory of generalized measures.

The four basic *t-norms* are the minimum $T_M = \min(x, y)$, the product $T_P = x \cdot y$, the Łukasiewicz *t-norm* $T_L = \max(x + y - 1, 0)$ and the drastic product T_D given by $T_D(1, x) =$

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$T_D(x, 1) = x$, and $T_D(x, y) = 0$ otherwise. It is well known that T_M is the strongest *t-norm* while T_D is the weakest one, i.e., for any triangular norm T we have $T_D \leq T \leq T_M$.

The construction of *t-norms* is very important in the theory of triangular norms. There are many methods concerning the construction of *t-norms*, e.g., the constructions based on the pseudo-inverse [1,2], on the additive generator [2,3], and on ordinal sums [2,4].

Recently, several new operations on [0, 1] which generalize the concept of triangular norm were introduced, e.g., the *uninorm* [5] and the *nullnorm* [6], the *triangular subnorm* (*t-subnorm* for short) [7,8], the boundary weak triangular norm (*bwt-norm* for short) [9].

A *t-subnorm* is an operation on [0, 1] which is commutative, associative, non-decreasing and bounded by the strongest triangular norm T_M. Obviously, a *t-norm* is always a *t-subnorm* but the converse is not true. A *bwt-norm* T is exactly a *t-subnorm* with $T(1, 1) = 1$; therefore, a *t-norm* is also a *bwt-norm* but not the other way round.

In [8], the author has investigated the properties of continuous *t-subnorms* and the construction of continuous Archimedean *t-subnorms* through the additive generators. In the light of these results, we shall consider the construction of *bwt-norms* through their additive generators. We are mainly interested in those proper *bwt-norms* which are not *t-norms*. Obviously, they satisfy the following conditions:

(i) $T(x, y) = T(y, x)$,
(ii) $T(T(x, y), z) = T(x, T(y, z))$,
(iii) $T(x, y) \leq T(x, z)$ if $y \leq z$,
(iv) $T(x, y) \leq \min(x, y)$ and $\exists\, x \in (0, 1)$ such that $T(x, 1) < x$.

2. Continuous additive generators

In [8], it was shown that a continuous *t-subnorm* T is proper if and only if $T(1, 1) < 1$. Observe that a proper *bwt-norm* T is a proper *t-subnorm* with $T(1, 1) = 1$; therefore any proper *bwt-norm* is not continuous. For a *t-norm*, a continuous additive generator always generates a continuous Archimedean *t-norm*. Proper *bwt-norms*, although they are discontinuous, may have continuous generators.

Theorem 2.1. *Let $f : [0, 1] \to [0, \infty]$ be a continuous non-increasing mapping and let $f(x) = 0$ if and only if $x = 1$. Then the operation $T : [0, 1]^2 \to [0, 1]$ given by*

$$T(x, y) = f^{(-1)}(f(x) + f(y)),$$

where $f^{(-1)} : [0, \infty] \to [0, 1]$ is the pseudo-inverse of f (i.e. $f^{(-1)}(y) = \sup\{x \in [0, 1] | f(x) > y\}$), is a left-continuous bwt-norm. f is called the additive generator of T.

Proof. Evidently, T is commutative and non-decreasing. For the associativity, observe that

$$T(T(x, y), z) = f^{(-1)}(f \circ f^{(-1)}(f(x) + f(y)) + f(z))$$
$$T(x, T(y, z)) = f^{(-1)}(f(x) + f \circ f^{(-1)}(f(y) + f(z))).$$

(i) If $f(x) + f(y) \notin Ran(f)$, then $f(x) + f(y) > f(0)$ and $f \circ f^{(-1)}(f(y) + f(z)) \geq f(y)$. Therefore $f(x) + f \circ f^{(-1)}(f(y) + f(z)) > f(0)$. Then $T(T(x, y), z) = 0 = T(x, T(y, z))$.
(ii) Similarly, if $f(y) + f(z) \notin Ran(f)$, then $T(x, T(y, z)) = 0 = T(T(x, y), z)$.