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Nonlocal boundary problems for a third-order one-dimensional nonlinear pseudoparabolic equation

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Abstract

We consider initial boundary value problems for a third-order nonlinear pseudoparabolic equation with one space dimension. The boundary condition is given by an integral; the function involved could exhibit singularities, which distinguishes this nonlocal condition from its Dirichlet or Neumann counterparts. By means of appropriate elliptic estimates we are able to seek solutions not only in the weighted spaces but also in the usual Sobolev spaces. The procedure is carried out in a unified way. Our results characterize a regularity of the pseudoparabolic operator that is different from that of the parabolic operator.

MSC: 35K70

Keywords: Pseudoparabolic equation; Elliptic estimate; Nonlocal boundary condition

1. Introduction

In this paper we consider the one-dimensional nonlinear pseudoparabolic equation

$$u_t - (a(x,t)u_{xt})_x = h(x,t,u,u_x,u_{xx}), \qquad \alpha < x < \beta,\ 0 < t < T \tag{1.1}$$

with the initial condition

$$u(x,0) = u_0(x), \qquad \alpha \le x \le \beta. \tag{1.2}$$

Two types of nonlocal boundary conditions will be studied:

$$u(\alpha,t) = 0, \qquad \int_\alpha^\beta u(x,t)\,\mathrm{d}x = 0, \qquad 0 \le t \le T \tag{1.3}$$

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and

$$\int_\alpha^\beta u(x,t)\,\mathrm{d}x = 0, \qquad \int_\alpha^\beta xu(x,t)\,\mathrm{d}x = 0, \qquad 0 \le t \le T. \tag{1.4}$$

We introduce some function spaces needed in this paper. Let X be a real Banach space and let $u : (0,T) \to X$ be an abstract function. We denote by $L^2(0,T;X)$ the standard Banach space with the norm

$$\|u\|_{L^2(0,T;X)} = \left(\int_0^T \|u(t)\|_X^2\,\mathrm{d}t\right)^{1/2}$$

and by $H^1(0,T;X)$ the Banach space of all $u \in L^2(0,T;X)$ with $u_t \in L^2(0,T;X)$. The norm in $H^1(0,T;X)$ is

$$\|u\|_{H^1(0,T;X)} = \left(\|u\|^2_{L^2(0,T;X)} + \|u_t\|^2_{L^2(0,T;X)}\right)^{1/2}.$$

Let $W^{1,\infty}(\alpha,\beta)$, $H^i(\alpha,\beta)$ $(i = 1,2,3)$ and $H^0(\alpha,\beta) = L^2(\alpha,\beta)$ be the standard Sobolev spaces [1]. Their norms are denoted by $\|\cdot\|_{1,\infty}$, $\|\cdot\|_i$ and $\|\cdot\|_0$, respectively.

We denote by $L^2_{wt}(\alpha,\beta)$ the weighted L^2-space with inner product

$$\langle u,v\rangle_{wt} = \int_\alpha^\beta (\beta - x)u(x)v(x)\,\mathrm{d}x$$

and by $H^i_{wt}(\alpha,\beta)$ $(i = 1,2)$ the weighted Sobolev space with inner product

$$\langle u,v\rangle_{i,wt} = \sum_{j=0}^{i} \langle u^{(j)}, v^{(j)}\rangle_{wt},$$

where $u^{(j)}$ is the jth derivative of u and $u^{(0)} = u$.

For the relationship of these spaces, we have [12, Lemma 2.1]

$$H^2_{wt}(\alpha,\beta) \subset H^1(\alpha,\beta) \subset H^1_{wt}(\alpha,\beta) \subset L^2(\alpha,\beta) \subset L^2_{wt}(\alpha,\beta).$$

We shall assume that:

(H_1) $a \in L^\infty(0,T;W^{1,\infty}(\alpha,\beta))$ and there exist positive constants a_1 and a_2 such that

$$0 < a_1 \le a(x,t) \le a_2, \qquad \forall (x,t) \in [\alpha,\beta]\times[0,T].$$

(H_2) $h : [\alpha,\beta]\times[0,T]\times\mathbb{R}^3 \to \mathbb{R}$ and there exists a positive constant K such that

$$|h(x,t,y_1^1,y_2^1,y_3^1) - h(x,t,y_1^2,y_2^2,y_3^2)| \le K(|y_1^1 - y_1^2| + |y_2^1 - y_2^2| + |y_3^1 - y_3^2|),$$

for $x \in [\alpha,\beta]$, $t \in [0,T]$, $y_i^j \in \mathbb{R}$, $i = 1,2,3$, $j = 1,2$.

By using appropriate elliptic estimates we are able to seek solutions not only in the weighted spaces as in [4,5] but also in the usual Sobolev spaces. This procedure will be carried out in a unified way. The main results of this paper are the following two theorems.

Theorem 1.1. *Assume that the assumptions* (H_1) *and* (H_2) *are satisfied. If* $u_0 \in H^2(\alpha,\beta)$ *such that*

$$u_0(\alpha) = 0, \qquad \int_\alpha^\beta u_0(x)\,\mathrm{d}x = 0,$$