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Existence and uniqueness of pseudo-almost periodic solutions to semilinear differential equations and applications

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Abstract

This paper deals with the existence and uniqueness of pseudo-almost periodic solutions to the semilinear differential equations of the form

$$u'(t) = Au(t) + Bu(t) + f(t, u(t)), \tag{*}$$

where A, B are densely defined closed linear operators on a Hilbert space $\mathbb{H}$, and $f : \mathbb{R} \times \mathbb{H} \mapsto \mathbb{H}$ is a jointly continuous function. Using both the so-called method of the invariant subspaces for unbounded linear operators and the classical Banach fixed-point principle, the existence of a pseudo-almost periodic solution to (*) is obtained under some suitable assumptions. As applications, we examine the existence and uniqueness of pseudo-almost periodic solutions to some second-order hyperbolic equations.

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Keywords: Pseudo-almost periodic function; Almost periodic function; Existence and uniqueness of pseudo-almost periodic solution; Infinitesimal generator of a c_0-group; Invariant subspace; Reducing subspace; Method of the invariant subspaces; Algebraic sum of unbounded linear operators; Banach fixed-point principle

1. Introduction

Let $(\mathbb{H}, \|\cdot\|, \langle\cdot,\cdot\rangle)$ be a Hilbert space. In [6], the author had shown through the so-called method of the invariant subspaces [7,10] that under some suitable assumptions, every bounded

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solution to the abstract differential equation of the form

$$u'(t) = Au(t) + Bu(t) + g(t), \qquad t \in \mathbb{R}, \tag{1.1}$$

where A, B are densely defined closed linear operators on $\mathbb{H}$, and $g : \mathbb{R} \mapsto \mathbb{H}$ is a pseudo-almost periodic function, was *pseudo-almost periodic*.

In this paper, we consider the original problem which consists of studying the existence and uniqueness of pseudo-almost periodic solutions to the semilinear equations of the form

$$u'(t) = Au(t) + Bu(t) + f(t, u(t)), \qquad t \in \mathbb{R}, \tag{1.2}$$

where A, B are densely defined closed (possibly non-commuting) linear operators on $\mathbb{H}$, and $f : \mathbb{R} \times \mathbb{H} \mapsto \mathbb{H}$ is a jointly continuous function. Under some appropriate assumptions it will be shown that Eq. (1.2) has a unique pseudo-almost periodic solution. For that, we essentially combine the method of exponentially stable c_0-groups, the method of the invariant spaces for unbounded linear operators, and the Banach fixed-point principle.

The idea of using the method of the invariant subspaces to study bounded, existence, or existence and uniqueness of almost automorphic [9,8] or almost periodic (mild) solutions to differential equations is quite recent and due to Diagana and N'Guérékata [6,7,10,16] and the references therein. Such a method is very efficient and works smoothly within the framework of abstract differential equations, functional and integral differential equations, and partial differential equations involving the *algebraic sum* of possibly non-commuting unbounded linear operators.

The concept of pseudo-almost periodicity (p.a.p.) initiated by Zhang in [19–21] in the early 1990s is a natural generalization of the classical almost periodicity (a.p.). Thus this new concept is welcome for implementing another existing generalization of the almost periodicity, the so-called notion of asymptotically almost periodicity (a.a.p) due to Fréchet, see, e.g., [4,15], and [16]. Moreover, note that the concept of pseudo-almost periodicity is a special case of the well-known Besicovitch almost periodic functions of order 1 denoted by $B_1(\mathbb{R}, \mathbb{X})$ (see [17] for details).

For more on the concepts of almost periodicity, and pseudo-almost periodicity, and related issues, we refer the reader to [1–3,5,6,11–13,16–21], and others.

The existence and uniqueness of pseudo-almost periodic solutions to some differential equations have been of great interest for many mathematicians in the past few decades (see [1–3,13,19–21] and the references therein). Here, we go back to examining the existence and uniqueness of a pseudo-almost periodic solution to Eq. (1.2) using the above-mentioned techniques.

As applications, we will examine the existence and uniqueness of a pseudo-almost periodic solution to the hyperbolic semilinear differential equations of the form

$$u''(t) + 2B\,u'(t) + A\,u(t) = f(t, u(t)), \tag{1.3}$$

where A, B are densely defined closed (possibly unbounded) linear operators acting on a Hilbert space $\mathbb{H}$, and $f : \mathbb{R} \times \mathbb{H} \mapsto \mathbb{H}$ is jointly continuous. For that, we reduce Eq. (1.3) to a first-order semilinear equation of the form Eq. (1.2) as follows: Setting $v(t) = u'(t)$, Eq. (1.3) can be rewritten into the Hilbert space $\mathbb{H} \times \mathbb{H}$ as

$$\mathcal{U}'(t) = \mathcal{A}\mathcal{U}(t) + \mathcal{B}\mathcal{U}(t) + F(t, \mathcal{U}(t)), \tag{1.4}$$