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Analysis of symmetric matrix valued functions I

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Abstract

For any symmetric function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, one can define a corresponding function on the space of $n \times n$ real symmetric matrices by applying f to the eigenvalues of the spectral decomposition. We show that this matrix valued function inherits from f the properties of continuity, Lipschitz continuity, strict continuity, directional differentiability, Frechet differentiability, continuous differentiability.
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1. Introduction

There has been growing interest in the variational analysis of spectral functions. This growing trend is due to the following features. On one hand, spectral functions have important applications to some fundamental problems in applied mathematics such as semidefinite programs and engineering problems (see [10] for many such applications). On the other hand, efficient non-smooth analysis tools have only been available in the past few years (see [18]). The property of semismoothness, as introduced by Mifflin [12,13] for functionals and scalar valued functions and further extended by Qi and Sun [15] for vector valued functions, is of particular interest due to the key role it plays in the superlinear convergence analysis of certain generalized Newton methods [6,14,15]. Recent attention in research on semismoothness is on symmetric matrix valued functions which have important applications to semidefinite complementarity problems [2,3,5,16,19–21]. Our study is inspired by the recent progress on spectral functions [8, 9,16] and progress on symmetric matrix valued functions [2,3,5,19,20].

Let $\mathcal{S}$ be the space of real symmetric matrices. We denote by $\mathcal{O}$ the group of all real orthogonal matrices. For any $A \in \mathcal{S}$, its (repeated) eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ are real and it admits a spectral decomposition of the form

$$A = P\text{diag}[\lambda_1, \ldots, \lambda_n]P^{\mathrm{T}} \tag{1.1}$$

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for some $P \in \mathcal{O}$, where $\text{diag}[\lambda_1, \ldots, \lambda_n]$ is the diagonal matrix with its ith diagonal entry is λ_i. A spectral function is a composition of a symmetric function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and the eigenvalue function $\lambda(.) : \mathcal{S} \longrightarrow \mathbb{R}^n$, i.e., the spectral function $f \circ \lambda : \mathcal{S} \longrightarrow \mathbb{R}$ is defined by

$$(f \circ \lambda)(A) := f(\lambda(A)) \quad (A \in \mathcal{S}).$$

Now we define, corresponding to (1.1), for a symmetric function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, the function $\tilde{f} : \mathcal{S} \longrightarrow \mathcal{S}$, by

$$\tilde{f}(A) := P\text{diag}[(f_1 \circ \lambda)(A), \ldots, (f_n \circ \lambda)(A)]P^{\mathrm{T}} \quad (A \in \mathcal{S}).$$

In this paper, we study some non-smoothness properties of $\tilde{f}$. Indeed, we show that the properties of continuity, strict continuity, Lipschitz continuity, directional differentiability, differentiability and continuous differentiability are each inherited by $\tilde{f}$ from f.

2. Preliminaries

Let $\mathcal{S}$ be the space of real symmetric matrices. It is clear that $\mathcal{S}$ is a subspace of $\mathbb{R}^{n\times n}$ with dimension $n(n+1)/2$. We denote by $\mathcal{O}$ the group of all real orthogonal matrices.

For any $A \in \mathcal{S}$, its (repeated) eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ are real and it admits a spectral decomposition of the form

$$A = P\text{diag}[\lambda_1, \ldots, \lambda_n]P^{\mathrm{T}} := P\text{diag}[\lambda(A)]P^{\mathrm{T}} \tag{2.1}$$

for some $P \in \mathcal{O}$, where $\text{diag}[\lambda_1, \ldots, \lambda_n]$ is the diagonal matrix with its ith diagonal entry λ_i. Note that (2.1) is independent of the choice of $P \in \mathcal{O}$ (see [1,4]).

Let $\lambda(.) : \mathcal{S} \longrightarrow \mathbb{R}^n$ be the eigenvalue function such that the $\lambda_i(A)$, $i = 1, \ldots, n$, yield eigenvalues of A for any $A \in \mathcal{S}$ and are ordered in a non-increasing order, that is, $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$. A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is called symmetric if f is invariant under coordinate permutation, that is, $f(x) = f(Px)$ for any permutation matrix P and any $x \in \mathbb{R}^n$. A spectral function is a composition of a symmetric function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and the eigenvalue function $\lambda(.) : \mathcal{S} \longrightarrow \mathbb{R}^n$, i.e., the spectral function $f \circ \lambda : \mathcal{S} \longrightarrow \mathbb{R}$ is defined by

$$(f \circ \lambda)(A) := f(\lambda(A)) \quad (A \in \mathcal{S}).$$

Let $\mathcal{P}$ denote the set of all permutation matrices in $\mathbb{R}^{n\times n}$. For given $\lambda \in \mathbb{R}^n$, $\mathcal{P}_\lambda$ denotes the stabilizer of λ defined by

$$\mathcal{P}_\lambda := \{P \in \mathcal{P} : P\lambda = \lambda\}.$$

Definition 2.1. A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with

$$f(x) = (f_1(x), \ldots, f_n(x))^{\mathrm{T}} \quad (x \in \mathbb{R}^n),$$

is called symmetric if each f_i is a symmetric function ($i = 1, 2, \ldots, n$).

Definition 2.2. We say that a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ satisfies the property ($\mathcal{E}$) if

$$\lambda \in \mathbb{R}^n \quad \text{and} \quad P \in \mathcal{P}_\lambda, \quad \text{then } Pf(\lambda) = f(P\lambda).$$

Now we define, corresponding to (2.1), for each symmetric function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with

$$f(x) = (f_1(x), \ldots, f_n(x))^{\mathrm{T}} \quad (x \in \mathbb{R}^n),$$

which satisfies the property ($\mathcal{E}$), a function $\tilde{f} : \mathcal{S} \longrightarrow \mathcal{S}$, by

$$\tilde{f}(A) := P\text{diag}[(f_1 \circ \lambda)(A), \ldots, (f_n \circ \lambda)(A)]P^{\mathrm{T}} \quad (A \in \mathcal{S}). \tag{2.2}$$

Note that $\tilde{f}(A)$ is independent of the choice of $p \in \mathcal{O}$ and that it belongs to $\mathcal{S}$.

In this paper, we use the following notation: vectors in $\mathbb{R}^n$ are viewed as columns and capital letters such as A, B, etc. denote matrices in $\mathcal{S}$. For each $\lambda \in \mathbb{R}^n$, define $\text{diag}[\lambda] := \text{diag}[\lambda_1, \ldots, \lambda_n]$.

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