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Bifurcation of limit cycles and integrability conditions for 6-parameter families of polynomial vector fields of arbitrary degree

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Abstract

We characterize the center variety and the maximum order of any focus of the 6-parameter family of real planar polynomial vectors given, in complex notation, by $\dot{z} = \mathrm{i}z + Az^{n_1}\bar{z}^{j_1} + Bz^{n_2}\bar{z}^{j_2} + Cz^{n_3}\bar{z}^{j_3}$, where $A, B, C \in \mathbb{C} \setminus \{0\}$, $(n_1, j_1) \neq (n_2, j_2) \neq (n_3, j_3)$, $n_k + j_k > 1$ for $k = 1, 2, 3$ $n_1 + j_1 = n_2 + j_2 = n_3 + j_3$, $|1 - n_3 + j_3| = |1 - n_2 + j_2| \neq |1 - n_1 + j_1|$ and $(1 - n_1 + j_1)(1 - n_2 + j_2) \neq 0$.

Keywords: Center manifolds; Integrability condition; Quadratic polynomial vector fields

1. Introduction and statement of the main results

A planar real analytic differential system in complex notation, $z = x + \mathrm{i}y$, can be written as

$$\dot{z} = \mathrm{i}z + F(z, \bar{z}). \tag{1}$$

We are concerned mainly with the conditions under which the origin of (1) is a center, and also with the question of determining the number of limit cycles which can bifurcate from the origin. These are very related issues and their knowledge will provide the dynamical properties of (1). In its real formulation, systems like (1) are used to model various phenomena like the crystallization of agates (see [19,24]).

A usual method of looking for nondegenerate centers (i.e. having purely imaginary eigenvalues) of planar polynomial vector fields such as that in (1) is to calculate the successive coefficients v_{2i+1}, called the *Liapunov constants* of the polynomial vector field, of the return map of the vector field about the origin. When all the v_{2i+1} vanish, then the origin is a *center* (notion first introduced by Poincaré in [17]). The set of coefficients for which all the v_i vanish is called the *center variety* of the family of polynomial vector fields. By the Hilbert Basis Theorem, the center variety is an algebraic set. On the other hand, if there is an integer N satisfying $v_{2k+1} = 0$ for $k < N$ but $v_{2N+1} \neq 0$, then the origin is a *fine focus of order N*. The order of a focus is an invariant of the system. At most N limit cycles can bifurcate from a fine focus of order N under perturbation of the coefficients. The stability of the origin is determined by the sign of the first non-vanishing Liapunov coefficient.

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In general, is very difficult to study the centers and the focus, since to do it requires a good knowledge not only of the common zeros of the polynomials v_i, but also of the finite generated ideal that they generate in the ring of polynomials, taking as variables the coefficients of the polynomial vector field. Furthermore, in general the calculation of the Liapunov constants is not easy, and the computational complexity of finding their common zeros grows very quickly. A number of algorithms have been developed to compute them automatically up to a certain order (see [2,5, 6,8,12–15] and the references therein). For analytical computation of the Liapunov constants, we are using the results of [3]. First, for each m, we will see which are the admissible monomials that appear in v_{2m+1}. After this, we will compute the coefficients of the polynomial v_{2m+1}. For doing that, we will use the knowledge of some types of centers for the systems (2).

The classification of centers in polynomial vector fields started with the quadratic ones with the works of Dulac, Kapteyn, Bautin, Żołądek,... see [20] for references. It continued with the symmetric cubic systems (those without quadratic terms) and projective quadratic systems [21,25,16]. Liapunov constants are also well known for Liénard systems [26,1,3].

Our goal is to study the centers and the maximum order of any focus of a 6-parameter family of linear polynomial vector fields with arbitrary homogeneous nonlinearities. Despite the fact that linear systems with quadratic and cubic nonlinearities are well understood (see [15,18,25]), very few families of centers of arbitrary degree are known; see for instance [4,9] for the case of 4-parameter families of polynomial vector fields of arbitrary degree, [10] for the case of 6-parameter families of polynomial vector fields, [23] for the case of an 8-parameter particular families of polynomial vector fields and [22] for the case of particular families of polynomial vector fields of arbitrarily numbers of parameters (greater or equal 6) and arbitrary degrees.

We consider the family of real polynomial differential equations in $(x, y) \in \mathbb{R}^2$ that in complex notation can be written as

$$\dot{z} = \mathrm{i}z + Az^{n_1}\bar{z}^{j_1} + Bz^{n_2}\bar{z}^{j_2} + Cz^{n_3}\bar{z}^{j_3}, \tag{2}$$

with $A, B, C \in \mathbb{C} \setminus \{0\}$, and where n_1, j_1, n_2, j_2, n_3 and j_3 are non-negative integers such that $(n_1, j_1) \neq (n_2, j_2) \neq (n_3, j_3)$, $n_k + j_k > 1$ for $k = 1, 2, 3$, $n_1 + j_1 = n_2 + j_2 = n_3 + j_3 = n$, and $|1 - n_3 + j_3| = |1 - n_2 + j_2| \neq |1 - n_1 + j_1|$. Note that this implies that system (2) is a linear system with homogeneous nonlinearities of degree $n_1 + j_1$, and

$$1 - n_3 + j_3 = -(1 - n_2 + j_2) = n_2 + j_2 - 1. \tag{3}$$

Set $N_1 = |1 - n_1 + j_1|$, $K_1 = |1 - n_2 + j_2|$, $L_1 = |1 - n_3 + j_3|$, $M_1 = \gcd\{N_1, K_1\}$, $N_1 = M_1 N_2$, $K_1 = M_1 K_2$.

We consider the case in which the parameters A, B, C are all different from zero, since the study of the center manifold variety in the case when at least one of the parameters is zero has already been studied in [9] (in the more general case of the nonlinearity of the vector field not necessarily being homogeneous). For an homogeneous nonlinearity, the results in [9] are:

Proposition 1. *For system (2) with $C = B = 0$ the following holds:*

(a) *if $j_1 \neq n_1 - 1$ then the center manifold variety of system (2) is the set $\{A \in \mathbb{C} \setminus \{0\}\}$,*
(b) *if $j_1 = n_1 - 1$ then the center manifold variety of system (2) is the set $\{A \in \mathbb{C} \setminus \{0\} : \mathrm{Re}(A) = 0\}$.*

The cases $A = B = 0$ or $A = C = 0$ are symmetric.

We set $M_{K_2,N_2} = (n - 1)(K_2 + N_2)/2$ and $d = (1 - n_1 - j_1)(1 - n_2 - j_2)$. We note that since $N_2, K_2 \geq 1$ and $\gcd\{N_2, K_2\} = 1$, $N_2 + K_2 \geq 3$. Furthermore, if $n_1 + j_1 - 1$ is odd, then N_1 and K_1 are odd, and thus, N_2 and K_2 are also odd. Hence, in this case, $N_2 + K_2$ is even. In short, M_{K_2,N_2} is always an integer number.

Proposition 2. *For system (2) with $C = 0$ the following holds:*

(a) *if $j_1 = n_1 - 1$ and $j_2 = n_2 - 1$, then the center manifold variety of system (2) is the set $\{(A, B) \in (\mathbb{C} \setminus \{0\})^2 : \mathrm{Re}(A) = \mathrm{Re}(B) = 0\}$,*
(b) *if $j_1 = n_1 - 1$ and $j_2 \neq n_2 - 1$, then the center manifold variety of system (2) is the set $\{(A, B) \in (\mathbb{C} \setminus \{0\})^2 : \mathrm{Re}(A) = 0\}$. The case $j_1 \neq n_1 - 1$ and $j_2 = n_2 - 1$ is symmetric,*
(c) *if $j_1 \neq n_1 - 1$ and $j_2 \neq n_2 - 1$, we have*
    (c.1) *$v_{2m+1} = 0$ for $m = 1, 2, \ldots, M_{K_2,N_2} - 1$,*