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Stability in functional differential equations established using fixed point theory[☆]

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Abstract

In this paper we consider a nonlinear scalar delay differential equation with variable delays and give some new conditions for the boundedness and stability by means of Krasnoselskii's fixed point theory. A stability theorem with a necessary and sufficient condition is proved. The results in [T.A. Burton, Stability by fixed point theory or Liapunov's theory: A comparison, Fixed Point Theory 4 (2003) 15–32; T.A. Burton, T. Furumochi, Asymptotic behavior of solutions of functional differential equations by fixed point theorems, Dynamic Systems and Applications 11 (2002) 499–519; B. Zhang, Fixed points and stability in differential equations with variable delays, Nonlinear Analysis 63 (2005) e233–e242] are improved and generalized. Some examples are given to illustrate our theory.

MSC: primary 34K20; 34K40

Keywords: Fixed points; Stability; Delay differential equations; Variable delays

1. Introduction

It has long been our view that Lyapunov's direct method is the leading technique for dealing with stability problems in many areas of differential equations. Yet, numerous difficulties with the theory and application to specific problems are encountered. Recently, Burton, Furumochi and others explored the use of fixed point theory in meeting some of those problems [1–10]. While Lyapunov's direct method usually requires pointwise conditions, the stability result established using fixed point theory calls for conditions which are average.

In this paper we consider the nonlinear scalar delay differential equation

$$x'(t) = -a(t)x(t - r_1(t)) + b(t)x^{1/3}(t - r_2(t)), \tag{1.1}$$

where $a, b \in C(R^+, R)$ and $r_1 \in C(R^+, R^+)$ with $t - r_1(t) \to \infty$ as $t \to \infty$, $r_2 \in C(R^+, [0, \gamma])$ for $\gamma > 0$.

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Many authors have investigated the special cases of Eq. (1.1). For example, Burton in [7] has investigated the boundedness and stability by applying Krasnoselskii's fixed point theory for the following equation:

$$x'(t) = -a(t)x(t - r_1) + b(t)x^{1/3}(t - r_2(t)), \tag{1.2}$$

where $r_1 \geq 0$ is a constant, and $a \in C(R^+, (0, \infty))$. Burton in [3] and Zhang in [10] have studied the following linear equation:

$$x'(t) = -a(t)x(t - r_1(t)). \tag{1.3}$$

Their main results are the following.

Theorem A (*Burton [7]*). *Suppose that there are constants* $0 < \beta < 1$ *and* $K > 0$ *satisfying*

$$\sup_{t\geq 0} |b(t)/a(t + r_1)| \leq \beta \tag{1.4}$$

and for $|t_1 - t_2| \leq 1$,

$$\left| \int_{t_1}^{t_2} a(u + r_1)\mathrm{d}u \right| \leq K|t_1 - t_2|, \tag{1.5}$$

while for $t \geq 0$

$$2 \sup_{t\geq 0} \int_{t-r_1}^{t} a(u + r_1)\mathrm{d}u + \beta < 1. \tag{1.6}$$

If ψ *is a given continuous initial function which is sufficiently small, then there is a solution* $x(t, 0, \psi)$ *of* (1.2) *on* R^+ *with* $|x(t, 0, \psi)| < 1$.

Moreover, for when (1.4) becomes: There is a $\beta < 1$ such that

$$\sup_{t\geq 0} |b(t)/a(t + r_1)| \leq \beta \quad \text{and} \quad b(t)/a(t + r_1) \to 0 \quad \text{as } t \to \infty, \tag{1.7}$$

the following stability result has also been obtained.

Theorem B (*Burton [7]*). *Let* (1.5)–(1.7) *hold and assume that* $\int_0^\infty a(s)\mathrm{d}s = \infty$. *If* ψ *is a given continuous initial function which is sufficiently small, then* (1.2) *has a solution* $x(t, \psi, 0) \to 0$ *as* $t \to \infty$.

Theorem C (*Burton [3]*). *Suppose that* $r_1(t) = r_1$, *a constant, and there exists a constant* $\alpha < 1$ *such that*

$$\int_{t-r_1}^{t} |a(s + r_1)|\mathrm{d}s + \int_0^t |a(s + r_1)|\mathrm{e}^{-\int_s^t a(u+r_1)\mathrm{d}u} \int_{s-r_1}^{s} |a(u + r_1)|\mathrm{d}u\mathrm{d}s \leq \alpha$$

for all $t \geq 0$ *and* $\int_0^\infty a(s)\mathrm{d}s = \infty$. *Then for every continuous initial function* $\psi : [-r_1, 0] \to R$, *the solution* $x(t) = x(t, 0, \psi)$ *of* (1.3) *is bounded and tends to zero as* $t \to \infty$.

Theorem D (*Zhang [10]*). *Suppose that* $r_1(t)$ *is differentiable, the inverse function* $q(t)$ *of* $t - r_1(t)$ *exists, and there exists a constant* $\alpha \in (0, 1)$ *such that for* $t \geq 0$

$$\liminf_{t\to\infty} \int_0^t a(q(s))\mathrm{d}s > -\infty,$$

$$\int_{t-r_1(t)}^{t} |a(q(s))|\mathrm{d}s + \int_0^t \mathrm{e}^{-\int_s^t a(q(u))\mathrm{d}u}|a(q(s))| \int_{s-r_1(s)}^{s} |a(q(v))|\mathrm{d}v\mathrm{d}s + \int_0^t \mathrm{e}^{-\int_s^t a(q(u))\mathrm{d}u}|a(s)||r_1'(s)|\mathrm{d}s \leq \alpha.$$

Then the zero solution of (1.3) *is asymptotically stable if and only if*

$$\int_0^t a(q(s))\mathrm{d}s \to \infty \quad \text{as } t \to \infty.$$