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Multipoint boundary value problems for nonlinear ordinary differential equations

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Abstract

In this paper we provide sufficient conditions for the existence of solutions to multipoint boundary value problems for nonlinear ordinary differential equations. We consider the case where the solution space of the associated linear homogeneous boundary value problem is less than 2. When this solution space is trivial, we establish existence results via the Schauder Fixed Point Theorem. In the resonance case, we use a projection scheme to provide criteria for the solvability of our nonlinear boundary value problem. We accomplish this by analyzing a link between the behavior of the nonlinearity and the solution set of the associated linear homogeneous boundary value problem.

MSC: 34B10; 34B15

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1. Introduction

In this paper we consider nonlinear ordinary differential equations subject to multipoint constraints. We provide criteria for the existence of solutions to boundary value problems of the form

$$y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_0(t)y(t) = g(y(t)); \quad 0 \le t \le 1 \tag{1}$$

subject to

$$\sum_{j=1}^{n} b_{ij}(0)y^{(j-1)}(0) + \sum_{j=1}^{n} b_{ij}(1)y^{(j-1)}(t_1) + \cdots + \sum_{j=1}^{n} b_{ij}(N)y^{(j-1)}(t_N) = 0 \tag{2}$$

for $i = 1, 2, \ldots n$.

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The points $t_k; k = 0, 1, \ldots, N$ are fixed with $0 = t_0 < t_1 < \cdots < t_N = 1$, and each coefficient b_{ij} is real. Throughout the paper it will be assumed that g and the coefficient functions $a_0, a_1, \ldots, a_{n-1}$ are all real valued, continuous and defined on $\mathbb{R}$, with $a_0(t) \neq 0$ for all $0 \leq t \leq 1$.

The conditions we establish for the solvability of the nonlinear boundary value problem depend on properties of the nonlinearity g, and on the solution space of the linear homogeneous boundary value problem

$$y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_0(t)y(t) = 0; \quad 0 \leq t \leq 1 \tag{3}$$

subject to the same boundary conditions found in Eq. (2).

Throughout our discussion we will be concerned with the case where the solution space of this linear homogeneous problem has dimension less than two. The dimension of this space plays a significant role in the complexity of the problem, and as a consequence, in the way the problem is approached. When the only solution of (3) and (2) is the trivial one, we analyze the nonlinear boundary value problem (1) and (2) using a relatively direct fixed point argument. When the solution space of (3) and (2) is one dimensional, the analysis is significantly more delicate. In this case we establish criteria for the existence of solutions to the nonlinear problem using a projection scheme [1–4,6,7,9,15,16] together with the Schauder Fixed Point Theorem.

2. Preliminaries

We formulate our boundary value problem in system form. In order to do so we introduce some notation.

The n by n matrix $A(t)$ is given by

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0(t) & -a_1(t) & -a_2(t) & \ldots & -a_{n-1}(t) \end{bmatrix},$$

and B_k is the n by n matrix given by $B_k = (b_{ij}(k))$. The map $f : \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$f(x) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g(x_1) \end{bmatrix}, \quad \text{where } x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ 0 \\ x_n \end{bmatrix}.$$

If v is an element of $\mathbb{R}^n$, we will use $|v|$ to denote the Euclidean norm of v.

It is clear that the nonlinear boundary problem (1) and (2) is equivalent to

$$x'(t) = A(t)x(t) + f(x(t)); \quad 0 \leq t \leq 1 \tag{4}$$

subject to

$$B_0x(0) + B_1x(t_1) + \cdots B_{N-1}x(t_{N-1}) + B_Nx(1) = 0, \tag{5}$$

where $x = \begin{bmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{bmatrix}$.

Similarly, the homogeneous boundary value problem (3) and (2) is equivalent to

$$x'(t) = A(t)x(t); \quad 0 \leq t \leq 1 \tag{6}$$

subject to the boundary conditions (5).

We will also need to analyze the solvability of the linear, nonhomogeneous boundary value problem

$$x'(t) = A(t)x(t) + h(t); \quad 0 \leq t \leq 1 \tag{7}$$

subject to the boundary conditions (5).