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Basic qualitative and quantitative results for solutions to nonlinear, dynamic equations on time scales with an application to economic modelling

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Abstract

This article investigates both basic qualitative and basic quantitative properties of solutions to first- and higher-order dynamic equations on time scales and thus provides a foundation and framework for future advanced nonlinear studies in the field. Particular focus lies in the: existence; uniqueness; dependency; approximation; and explicit representation, of solutions to nonlinear initial value problems. The main tools used are from modern areas of nonlinear analysis, including: the fixed-point theorems of Banach and Schäfer; the method of successive approximations; a novel definition of measuring distance in metric spaces and normed spaces; and a "separation" of variables technique is introduced to the general time scale setting.

The new results compliment and extend those of Stefan Hilger's seminal paper of 1990.

As an application of the new results we present and analyse a simple model from economics, known as the Keynesian–Cross model with "lagged" income, in the general time scale environment.

Ideas suggesting further applications and possible new directions for the novel results are also presented.

MSC: 39A10; 39A12

Keywords: Existence and uniqueness of solutions; Approximation of solutions; Fixed-point methods; Initial value problems; Time scales; Separation of variables

1. Introduction

Important advancements in all the physical, life and social sciences rest heavily on the existence of a mathematical framework to describe, to solve and to better understand the problems from these fields. Historically, two separate approaches have dominated mathematical modelling: the field of differential equations, termed "continuous dynamic modelling", where variables (e.g. time) are assumed to flow in a continuous fashion; and the area of difference equations, termed "discrete dynamic modelling", where variables (such as time) are assumed to vary in a discrete manner.

Traditionally, researchers have assumed that dynamical processes are *either* continuous *or* discrete and thus have employed *either* differential equations *or* difference equations – but not elements from both schools of thought – for

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the mathematical description and analysis of dynamic models. For example, the classical approach of Domar [3] used differential equations to analyse an expanding economy, while, on the other hand, Harrod's method [3] of mathematical description for economic growth involved the field of difference equations.

This blanket assumption that processes are either solely continuous or solely discrete, while convenient for traditional mathematical approaches, is flawed because, in reality, many processes do feature both continuous and discrete elements. Thus, traditional mathematical modelling techniques, such as differential or difference equations, provide a limited understanding of these types of physical models and appears to be a case of modifying the assumptions on a physical problem to best fit the mathematics, rather than vice versa.

In particular, certain economically important phenomena do not possess solely continuous properties or solely discrete aspects. Rather, these phenomena contain processes that feature elements of *both* the continuous *and* the discrete. A simple example of this hybrid continuous–discrete behaviour is seen in "seasonally breeding populations in which generations do not overlap. Many natural populations, particularly among temperate-zone insects (including many economically important crop and orchid pests) are of this kind" [30, p.460]. These insects lay their eggs just before the generation dies out at the end of the season, with the eggs laying dormant, hatching at the start of the next season giving rise to a new, nonoverlapping generation. The continuous–discrete behaviour in seen in the fact that during each generation the population varies continuously (due to mortality, resource consumption, predation, interaction etc.), while the population varies in a discrete fashion between the end of one generation and the beginning of the next [17, p.620].

In addition, continuous–discrete processes are seen in: robust 3D tracking in shape and motion estimation [31, p. 712] ; option-pricing and stock dynamics in finance [6, p. 3]; the frequency of markets and duration of market trading in economics [24, p. 1], [28, p. 2]; large-scale models of DNA dynamics [26, p. 2504]; gene mutation fixation [9, pp. 1–2] ; and "hybrid systems" where stop–start elements are naturally seen.

Current approaches, such as the field of differential equations or the field of difference equations are ill-equipped as separate fields to accurately describe the above models because these mathematical areas are limited to either the continuous or the discrete and thus are of limited value in understanding these models. Therefore, there is a great need to find a more flexible mathematical framework to accurately model the aforementioned dynamical blend of systems so that they are precisely described, better-understood and significant advancements are made.

To address the aforementioned needs, an emerging, progressive and modern area of mathematics, known as the field of *dynamic equations on time scales*, has the capacity to act as the framework to effectively describe the above phenomena and to make advancements in their associated fields. Created by Hilger in 1990 [23] and developed by others (see [5,27] and references therein), this new and exciting type of mathematics is more general and versatile than the traditional theories of differential and difference equations as it can, under one framework, mathematically describe continuous–discrete hybrid processes and hence is the optimal way forward for accurate and malleable mathematical modelling. In fact, the progressive field of dynamic equations on time scales contains, links and extends the classical theory of differential and difference equations.

Much of the "linear" theory of dynamic equations on time scales has been presented in [5], however, there is significantly less literature available on the basic "nonlinear" theory of the field. It is important to bridge this gap between known linear studies and unknown nonlinear theory, as the processes in our world are inherently nonlinear and such investigations will provide an important platform for gaining a deeper understanding of our environment.

This paper considers first-order dynamic equations of the type

$$\mathbf{x}^{\Delta} = \mathbf{f}(t, \mathbf{x}), \quad t \in [a, b]_{\mathbb{T}} := [a, b] \cap \mathbb{T}; \tag{1.1}$$

$$\mathbf{x}^{\Delta} = \mathbf{f}(t, \mathbf{x}^{\sigma}), \quad t \in [a, b]_{\mathbb{T}}; \tag{1.2}$$

subject to the initial condition

$$\mathbf{x}(a) = \mathbf{A}; \tag{1.3}$$

where $\mathbf{f} : [a, b]_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ may be a nonlinear function, $n \geq 1$; t is from a so-called "time scale" $\mathbb{T}$ (which is a nonempty closed subset of $\mathbb{R}$); $\mathbf{x}^{\Delta}$ is the generalised "delta" derivative of $\mathbf{x}$; and $a < b$ are given constants in $\mathbb{T}$; and $\mathbf{A}$ is a given constant in $\mathbb{R}^n$. Eq. (1.1) subject to (1.3) is known as a dynamic initial value problem (IVP) on time scales. Eqs. (1.2) and (1.3) are defined similarly, where $\mathbf{x}^{\sigma} := \mathbf{x} \circ \sigma$ with σ a function to be defined a little later.