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Positive solutions of nonresonance semipositone singular Dirichlet boundary value problems

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Abstract

In the case where the nonlinearity term is allowed to change sign, we study the nonresonance semipositone singular Dirichlet boundary value problem (BVP)

$$\begin{cases} -x'' + \rho p(t)x = \lambda[f(t,x) + g(t,x)], & 0 < t < 1, \\ x(0) = x(1) = 0, \end{cases}$$

where $\lambda > 0$ is a parameter, $\rho > 0$ is a constant. We derive an interval of λ such that for any λ lying in this interval, the semipositone BVP has at least one positive solution if f is superlinear or sublinear. The results obtained improve and extend many recent results. Our approach is based on Krasnaselskii's fixed point theorem in cones.

MSC: 34B15; 34B25

Keywords: Singular boundary value problems; Semipositone; Positive solutions; Nonresonance; Superlinear; Sublinear

1. Introduction

The singular boundary value problem

$$\begin{cases} -x'' = \lambda f(t,x), & 0 < t < 1, \\ x(0) = x(1) = 0, \end{cases} \tag{1.1}$$

arises from many branches of applied mathematics and physics such as gas dynamics, Newtonian fluid mechanics, nuclear physics and has been widely studied by many authors (see [1–5]).

Recently, there is much attention focused on the existence of positive solutions for the more general cases. For example, Wei [6] and Pang [7] considered the nonresonance singular boundary value problem

$$\begin{cases} -x'' + \rho p(t)x = f(t,x), & 0 < t < 1, \\ x(0) = x(1) = 0, \end{cases} \tag{1.2}$$

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where $\rho > 0$ and for this case

$$\begin{cases} -x'' + \rho p(t)x = 0, & 0 < t < 1, \\ x(0) = x(1) = 0, \end{cases}$$

has only trivial solutions. By using the upper and lower solution method and the fixed point theorem, Wei and Pang showed that the BVP (1.2) has at least one positive solution when f is superlinear or sublinear. But by a general analysis, it is not difficult to find that most papers require that the nonlinearity term is nonnegative, and for details, see [1–7]. In this paper, we shall consider the following semipositone boundary value problem

$$\begin{cases} -x'' + \rho p(t)x = \lambda[f(t,x) + g(t,x)], & 0 < t < 1, \\ x(0) = x(1) = 0, \end{cases} \tag{1.3}$$

where $\lambda > 0$ is a parameter, $\rho > 0$ is a constant, f and g may be singular at $t = 0, 1$. Moreover, the nonlinearity term is allowed to change sign. These types of problems are referred to as semipositone problems in the literature and they arise naturally in chemical reactor theory [9]. The constant λ is usually called the Thiele modulus, and in applications one is interested in showing the existence of positive solutions for small positive λ values. However, because the solution of the semipositone problem does not possess concavity, it is usually very difficult to seek for a positive solution of this type of problems [8,10,11]. In this paper, we, using Krasnaselskii's fixed point theorem in cones and combining with an available transformation, obtain an interval of λ which ensures the existence of at least one positive solution for the BVP (1.3). The emphasis here is that our technique can also be used to solve other types of semipositione boundary value problems.

This paper is organized as follows. In Section 2, we present some lemmas that will be used to prove our main results. In Section 3, by using Krasnaselskii's fixed point theorem in cones we discuss the existence of positive solutions of the BVP (1.3). In each theorem and corollary, an interval of eigenvalues is determined to ensure the existence of positive solutions for the BVP (1.3).

2. Preliminaries and some lemmas

Firstly, let us list the following assumptions that are used throughout the paper:

(H$_1$) $p(t) \in C(0,1)$, $p(t) \geq 0$, $t \in (0,1)$ and $\int_0^1 t(1-t)p(t)\mathrm{d}t < +\infty$.

(H$_2$) $f(t,x) \in C((0,1) \times [0,+\infty), [0,+\infty))$, and there exist constants $\lambda_1 \geq \lambda_2 > 1$ such that, for any $t \in (0,1)$, $x \in [0,+\infty)$,

$$c^{\lambda_1} f(t,x) \leq f(t,cx) \leq c^{\lambda_2} f(t,x), \quad \forall\, c \in [0,1]. \tag{2.1}$$

(H$_2^*$) $f(t,x) \in C([0,1] \times [0,+\infty), [0,+\infty))$, and there exist constants $0 < \lambda_3 \leq \lambda_4 < 1$, such that, for any $t \in [0,1]$, $x \in [0,+\infty)$,

$$c^{\lambda_4} f(t,x) \leq f(t,cx) \leq c^{\lambda_3} f(t,x), \quad \forall\, c \in [0,1]. \tag{2.2}$$

(H$_3$) $g(t,x) \in C((0,1), (-\infty,+\infty))$, further, for any $t \in (0,1)$, $x \in [0,+\infty)$, there exists a function $q(t) \in L^1((0,1),(0,+\infty))$ such that $|g(t,x)| \leq q(t)$.

(H$_3^*$) $g(t,x) \in C([0,1], (-\infty,+\infty))$, further, for any $t \in [0,1]$, $x \in [0,+\infty)$, there exists a function $q(t) \in C([0,1],(0,+\infty))$ such that $|g(t,x)| \leq q(t)$.

(H$_4$)

$$0 < \int_0^1 t(1-t)[f(t,1) + q(t)]\mathrm{d}t < +\infty.$$

Remark 2.1. For any $c \geq 1$, $(t,x) \in (0,1) \times [0,+\infty)$, we have

$$c^{\lambda_2} f(t,x) \leq f(t,cx) \leq c^{\lambda_1} f(t,x) \tag{2.3}$$

$$c^{\lambda_3} f(t,x) \leq f(t,cx) \leq c^{\lambda_4} f(t,x). \tag{2.4}$$