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Regularity of viscosity solutions to a degenerate nonlinear Cauchy problem☆

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Abstract

This paper is concerned with the regularity of certain weak solutions to the Cauchy problem in $\mathbb{R}^N$ for a degenerate nonlinear parabolic equation. We get the Hölder regularity of the weak solution when the parameter $\gamma \geq \sqrt{N-1}$ and improve the parameter's region from $\gamma \geq \sqrt{2N}-1$ to $\gamma \geq \sqrt{N-1}$ since $\sqrt{N-1} < \sqrt{2N}-1$.

MSC: 35K55; 35K65; 35D10; 35K15

Keywords: Viscosity solution; Degenerate; Hölder regularity

1. Introduction

Define $\Omega = \mathbb{R}^N \times \mathbb{R}^+$ and let us consider the Cauchy problem

$$\begin{cases} u_t = u\triangle u - \gamma|\nabla u|^2, & (x,t) \in \Omega \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N \end{cases} \tag{1}$$

where γ is a nonnegative constant and u_0 is a bounded continuous and nonnegative function on R^N.

Problem (1) becomes degenerate at the points where u vanishes. Therefore, in general, it has no classical solutions and we have to consider its weak solutions.

The weak solution is defined as follows:

Definition 1.1. A function $u \in L^\infty(\Omega) \cap L^2_{\text{loc}}([0,+\infty); H^1_{\text{loc}}(\mathbb{R}^N))$ is called a weak solution of problem (1) if $u \geq 0$ a.e. in Ω and for all $T > 0$, the integral equality

$$\int_{R^N} u_0\psi(0)\mathrm{d}x + \int_{\mathbb{R}^N\times(0,T)} u\frac{\partial\psi}{\partial t} - u\nabla u\cdot\nabla\psi - (1+\gamma)|\nabla u|^2\psi\,\mathrm{d}x\mathrm{d}t = 0$$

holds for any $\psi \in C^{1,1}(\mathbb{R}^N \times [0,T])$ with compact support in $\mathbb{R}^N \times [0,T]$.

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Let $w_\epsilon \geq 0$ be the classical solution of the problem

$$\begin{cases} u_t = u\triangle u - \gamma|\nabla u|^2 + \epsilon\triangle u, & (x,t) \in \Omega \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N \end{cases} \tag{2}$$

where $\gamma \geq 0$. Then $u_\epsilon = w_\epsilon + \epsilon > 0$ is the classical solution of the problem

$$\begin{cases} u_t = u\triangle u - \gamma|\nabla u|^2, & (x,t) \in \Omega \\ u(x,0) = u_0(x) + \epsilon, & x \in \mathbb{R}^N. \end{cases} \tag{3}$$

$u_\epsilon(x,t)$ is nonincreasing with respect to ϵ; thus

$$u(x,t) = \lim_{\epsilon\to 0} u_\epsilon(x,t)$$

is well defined in $\bar{\Omega}$. It has been proved by Bertsh, Passo and Ughi in [2] that u is a weak solution of problem (1). Because u_0 is bounded, using the maximum principle in problem (3), u_ϵ is bounded and $\{u_\epsilon\}_{\epsilon\to 0}$ is uniformly bounded.

Definition 1.2 (*[1]*). The weak solution that we got above is called a viscosity solution of problem (1).

It turns out that weak solutions of problem (1) are not uniquely determined and there are solutions other than the viscosity solution. In the case $\gamma = 0$ this was shown by Ughi [4] and Passo [5]. For a general discussion of this nonuniqueness phenomenon we refer the reader to Bertsh, Passo and Ughi's paper [6].

Let $u(x,t)$ be the viscosity solution of problem (1). It has been shown in [2] that

$$\begin{aligned} &N = 1, \qquad \gamma > 0 \Longrightarrow u \in C^\infty(\bar{\Omega}) \\ &\gamma \geq \frac{1}{2}N \Longrightarrow u \in C(\bar{\Omega}) \cap C^{1,1}(\Omega) \\ &N \geq 2, \qquad 0 \leq \gamma < 1 \Longrightarrow u \text{ is not necessarily continuous in } \bar{\Omega}. \end{aligned}$$

In [3], Professors Lu and Qian had proved that $\gamma \geq \sqrt{2N} - 1 \Longrightarrow u$ is Lipschitz continuous in x and Hölder continuous in t. Therefore, the regularity of the weak solution depends heavily on the relation between parameter γ and the space dimension N.

In this paper, we improve the parameter's region from $\gamma \geq \sqrt{2N} - 1$ to $\gamma \geq \sqrt{N-1}$ since $\sqrt{N-1} < \sqrt{2N} - 1$. Also, we get the same result as in [3] but the region is $\gamma \geq \sqrt{N-1}$.

2. Lemmas

To get our main result, we begin with a general form of problem (1), i.e.,

$$\begin{cases} u_t = \alpha_1 u^{\beta_1}\triangle u + \alpha_2 u^{\beta_2} w, & w = \dfrac{1}{2}|\nabla u|^2, \ (x,t) \in \Omega \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N \end{cases} \tag{4}$$

and firstly prove two lemmas for this problem.

Lemma 2.1. *If $\alpha_1 > 0$, $\beta_2 = \beta_1 - 1$, there exists a constant s such that*

$$2\alpha_2\beta_1 - 2\alpha_2 - s\alpha_2 + 2s(s+1)\alpha_1 + N\alpha_1\beta_1^2 \leq 0$$

and

$$|\nabla(u_0^{1+\frac{s}{2}})| \leq M$$

for a nonnegative constant M. Then the viscosity solution u of problem (4) satisfies $|\nabla(u^{1+\frac{s}{2}})| \leq M$ in $\bar{\Omega}$.