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Nonlinear Analysis 68 (2008) 794–803

Nonlinear Analysis

www.elsevier.com/locate/na

Stability of differential equations with piecewise constant arguments of generalized type

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Received 20 July 2006; accepted 17 November 2006

Abstract

In this paper we continue to consider differential equations with piecewise constant argument of generalized type (EPCAG) [M.U. Akhmet, Integral manifolds of differential equations with piecewise constant argument of generalized type, Nonlinear Anal. TMA 66 (2007) 367–383]. A deviating function of a new form is introduced. The linear and quasilinear systems are under discussion. The structure of the sets of solutions is specified. Necessary and sufficient conditions for stability of the zero solution are obtained. Our approach can be fruitfully applied to the investigation of stability, oscillations, controllability and many other problems of EPCAG. Some of the results were announced at The International Conference on Hybrid Systems and Applications, University of Louisiana, Lafayette, 2006.

MSC: 34A36; 34D20; 34A30

Keywords: Stability; Quasilinear systems; Piecewise constant argument of generalized type

1. Introduction and preliminaries

Let $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{R}$ be the sets of all integers, natural and real numbers, respectively. Denote by $\|\cdot\|$ the Euclidean norm in $\mathbb{R}^n$, $n \in \mathbb{N}$. Fix two real valued sequences θ_i, ζ_i, $i \in \mathbb{Z}$, such that $\theta_i < \theta_{i+1}, \theta_i \le \zeta_i < \theta_{i+1}$ for all $i \in \mathbb{Z}$, $|\theta_i| \to \infty$ as $|i| \to \infty$.

In the present paper we shall consider the following two equations:

$$z'(t) = A_0(t)z(t) + A_1(t)z(\gamma(t)), \tag{1}$$

and

$$z'(t) = A_0(t)z(t) + A_1(t)z(\gamma(t)) + f(t, z(t), z(\gamma(t))), \tag{2}$$

where $z \in \mathbb{R}^n$, $t \in \mathbb{R}$, $\gamma(t) = \zeta_i$, if $t \in [\theta_i, \theta_{i+1})$, $i \in \mathbb{Z}$.

The following assumptions will be needed throughout the paper:

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0362-546X/$ - see front matter
doi:10.1016/j.na.2006.11.037

(C1) $A_0, A_1 \in C(\mathbb{R})$ are $n \times n$ real valued matrices;
(C2) $f(t, x, y) \in C(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n)$ is an $n \times 1$ real valued function;
(C3) $f(t, x, y)$ satisfies the condition

$$\|f(t, x_1, y_1) - f(t, x_2, y_2)\| \le L(\|x_1 - x_2\| + \|y_1 - y_2\|), \tag{3}$$

where $L > 0$ is a constant, and the condition

$$f(t, 0, 0) = 0, \quad t \in \mathbb{R}; \tag{4}$$

(C4) matrices A_0, A_1 are uniformly bounded on $\mathbb{R}$;
(C5) $\inf_{\mathbb{R}} \|A_1(t)\| > 0$;
(C6) there exists a number $\bar{\theta} > 0$ such that $\theta_{i+1} - \theta_i \le \bar{\theta}$, $i \in \mathbb{Z}$;
(C7) there exists a number $\theta > 0$ such that $\theta_{i+1} - \theta_i \ge \theta$, $i \in \mathbb{Z}$;
(C8) there exists a positive real number p such that

$$\lim_{t\to\infty} \frac{i(t_0, t)}{t - t_0} = p$$

uniformly with respect to $t_0 \in \mathbb{R}$, where $i(t_0, t)$ denotes the number of points θ_i in the interval (t_0, t).

The theory of differential equations with piecewise constant argument (EPCA) of the type

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(t, x(t), x([t])), \tag{5}$$

or

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(t, x(t), x(2[(t + 1)/2])), \tag{6}$$

where $[\cdot]$ signifies the greatest integer function, was initiated in [5] and has been developed by many authors [1,4,6,10–17]. Applications of EPCA are discussed in [3,9,16]. They are hybrid equations, in that they combine the properties of both continuous systems and discrete equations. The novelty of our paper is that we find a class of the systems which in their properties are very close to ordinary differential equations. We believe that our proposals may stimulate new ideas advancing the theory and adding to the previous significant achievements in that direction.

The novel idea of our paper is that systems (1) and (2) are EPCA of general type (EPCAG). Indeed if we take $\zeta_i = \theta_i = i, i \in \mathbb{Z}$, then (1) takes the form of (5), and (1) and (2) take the form of (6) if $\theta_i = 2i - 1$, $\zeta_i = 2i$, $i \in \mathbb{Z}$. The particular case of EPCAG, when $\zeta_i = \theta_i$, $i \in \mathbb{Z}$, is considered in [2]. The existing method of investigation of EPCA, as proposed by its founders [5,16], is based on the reduction of EPCA to discrete equations. A new approach proposed in [2] is based on the construction of an equivalent integral equation. We consider the initial value problem in the general form, that is when t_0 is an arbitrary real number, not necessarily one of the moments θ_i.

One can easily see that Eqs. (1) and (2) have the form of functional differential equations:

$$z'(t) = A_0(t)z(t) + A_1(t)z(\zeta_i), \tag{7}$$

$$z'(t) = A_0(t)z(t) + A_1(t)z(\zeta_i) + f(t, z(t), z(\zeta_i)), \tag{8}$$

respectively, if $t \in [\theta_i, \theta_{i+1})$, $i \in \mathbb{Z}$.

That is, these systems have the structure of a continuous dynamical system within the intervals $[\theta_i, \theta_{i+1})$, $i \in \mathbb{Z}$.

In our paper we assume that the solutions of the equation are *continuous functions*. But the deviating function $\gamma(t)$ is discontinuous. Hence, in general, the right-hand sides of (1) and (2) have discontinuities at moments θ_i, $i \in \mathbb{Z}$. Summarizing, we consider the solutions of the equations as functions, which are continuous and continuously differentiable within intervals $[\theta_i, \theta_{i+1})$, $i \in \mathbb{Z}$.

We use the following definition, which is a version of a definition from [13], modified for our general case.

Definition 1.1. A continuous function $z(t)$ is a solution of (1) and (2) on $\mathbb{R}$ if:

(i) the derivative $z'(t)$ exists at each point $t \in \mathbb{R}$ with the possible exception of the points θ_i, $i \in \mathbb{Z}$, where the one-sided derivatives exist;
(ii) the equation is satisfied for $z(t)$ on each interval (θ_i, θ_{i+1}), $i \in \mathbb{Z}$, and it holds for the right derivative of $z(t)$ at the points θ_i, $i \in \mathbb{Z}$.