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Nonlinear Analysis 68 (2008) 804–827
Nonlinear Analysis
www.elsevier.com/locate/na

Fractional Sobolev spaces and topology

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Received 24 July 2006; accepted 17 November 2006

Abstract

Consider the Sobolev class $W^{s,p}(M,N)$ where M and N are compact manifolds, and $p \geq 1, s \in (0, 1+1/p)$. We present a necessary and sufficient condition for two maps u and v in $W^{s,p}(M,N)$ to be continuously connected in $W^{s,p}(M,N)$. We also discuss the problem of connecting a map $u \in W^{s,p}(M,N)$ to a smooth map $f \in C^\infty(M,N)$.

Keywords: Fractional Sobolev spaces between manifolds; Homotopy

1. Introduction

Let M and N be compact connected smooth boundaryless Riemannian manifolds. Throughout the paper we assume that they are isometrically embedded into $\mathbb{R}^a$ and $\mathbb{R}^l$ respectively and that $m := \dim M \geq 2$. Our functional framework is the Sobolev space

$$W^{s,p}(M,N) = \{u \in W^{s,p}(M,\mathbb{R}^l) : u(x) \in N \text{ a.e.}\},$$

with $1 \leq p < \infty, 0 < s$. The space $W^{s,p}(M,N)$ is equipped with the standard metric $\mathrm{d}(u,v) = \|u-v\|_{W^{s,p}}$. The main purpose of this paper is to determine whether or not $W^{s,p}(M,N)$ is path connected and if not, when two elements u and v in $W^{s,p}(M,N)$ can be continuously connected in $W^{s,p}(M,N)$; that is, when there exists $H \in C^0([0,1], W^{s,p}(M,N))$ such that $H(0) = u$ and $H(1) = v$. If this is the case, we say that 'u and v are $W^{s,p}$ connected' (or $W^{s,p}$ homotopic).

Homotopy theory in the framework of Sobolev spaces is essential when studying certain problems in the calculus of variations. This is the case when the admissible functions are defined on a manifold M into a manifold N. One may hope to find multiple minimizers to these problems, ideally one in each homotopy class (see [15,16] and also [3]).

The topology of $W^{s,p}(M,N)$ depends on two features of the problem, namely the topology of M and N, and the value of s and p. When $s = 1$, the study of the topology of $W^{1,p}(M,N)$ was initiated in [4]. The analysis of homotopy classes (for $s = 1$) was subsequently tackled in [9] (see also [15,16] for related and earlier results). These results have been generalized to $W^{s,p}(M,N)$ for non-integer values of s and $1 < p < \infty$ when M is a smooth, bounded, connected open set in a Euclidean space and when $N = S^1$ (see [5]). In this case, the proofs exploit in an essential way the fact that the target manifold is S^1. In contrast, our main concern is to determine to what extent

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0362-546X/$ - see front matter
doi:10.1016/j.na.2006.11.038

the methods of [9] and the tools of [4] can be adapted to the case $s \neq 1$. Throughout the paper, we assume that $0 < s < 1 + 1/p$ or $sp \geq \dim M$.

Our first result gives some conditions which imply that $W^{s,p}(M, N)$ is path connected:

Theorem 1. *Let $0 < s < 1 + 1/p$. Then the space $W^{s,p}(M, N)$ is path connected when $sp < 2$.*

When $s = 1$, this result was proved in [4], where the condition $p < 2$ (for $s = 1$) is seen to be sharp. For instance, $W^{1,2}(S^1 \times \Lambda, S^1)$, where Λ is any open connected set, is not path connected.

In the case $sp \geq 2$, we have:

Theorem 2. *Assume that $0 < s < 1 + 1/p, 2 \leq sp < \dim M$ and that there exists $k \in \mathbb{N}$ with $k \leq [sp] - 1$ such that $\pi_i(M) = 0$ for $1 \leq i \leq k$, $\pi_i(N) = 0$ for $k + 1 \leq i \leq [sp] - 1$. Then the space $W^{s,p}(M, N)$ is path connected.*

The case $s = 1$ of the above theorem is Corollary 1.1 in [9].

More generally, it is natural to compare the connected components of $W^{s,p}(M, N)$ to those of $C^0(M, N)$. In certain cases, this is indeed possible:

Theorem 3. (a) *If $sp \geq \dim M$ then $W^{s,p}(M, N)$ is path connected if and only if $C^0(M, N)$ is path connected.*

(b) *The $W^{s,p}$ homotopy classes are in bijection with the C^0 homotopy classes when $0 < s < 1 + 1/p, 2 \leq sp < \dim M$ and $\pi_i(N) = 0$ for $[sp] \leq i \leq \dim M$.*

Statement (a) is well known and can be proved as in the appendix of [4]. Part (b) for $s = 1$ was obtained in [9], Corollary 5.2.

When $s = 1$, Theorems 2 and 3 are particular cases of a more general result in [9] which asserts that there is a one-to-one map from the connected components of $W^{1,p}(M, N)$ into the connected components of $C^0(M^{[p]-1}, N)$. Here, $M^{[p]-1}$ denotes a $[p] - 1$ skeleton of M. This may be re-expressed as follows: two maps u and v in $W^{1,p}(M, N)$ are $W^{1,p}$ homotopic if and only if u is $[p]-1$ homotopic to v. For an accurate definition of $[p]-1$ homotopy, one should refer to [9] or to Section 6. Roughly speaking, this means that for a generic $[p] - 1$ skeleton $M^{[p]-1}$ of M, $u|_{M^{[p]-1}}$ and $v|_{M^{[p]-1}}$ are homotopic. This makes sense because for a generic $[p] - 1$ skeleton, u and v are both $W^{1,p}$ on these skeletons and hence continuous, by the Sobolev embedding. There is a corresponding version of this result in which $W^{1,p}$ is replaced by $W^{s,p}$:

Theorem 4. *Assume that $0 < s < 1 + 1/p, 2 \leq sp < \dim M$. Let $u, v \in W^{s,p}(M, N)$. Then u and v are $W^{s,p}$ connected if and only if u is $[sp] - 1$ homotopic to v.*

The techniques in [9] can be adapted in order to prove not only Theorem 4 but also the more general result where the condition $2 \leq sp < \dim M$ is replaced by: $0 < sp < \dim M$ and $sp \neq 1$. In turn, this last result implies Theorem 1 when $sp < 2, sp \neq 1$. However, the case $sp = 1$ seems delicate to handle via these techniques. This is why we give a proof of Theorem 1 based on the tools of [4]. Besides its independent interest, it turns out that the technical core of the proof of Theorem 1 is also the technical core of the proof of Theorem 4. Furthermore, the techniques in [4] are more likely to allow some extensions to the case $s > 1 + 1/p$.

Another strategy for showing that two elements in $W^{s,p}(M, N)$ are $W^{s,p}$ connected is based on the property $P(u)$ defined for any $u \in W^{s,p}(M, N)$ by:

($P(u)$) The map u is $W^{s,p}$ homotopic to some $\tilde{u} \in C^\infty(M, N)$.

We proceed to explain the interest of this property. Assume that $P(u)$ and $P(v)$ are true, where $u, v \in W^{s,p}(M, N)$, and that $\tilde{u}$ and $\tilde{v}$ are C^0 homotopic. So, there exists $F \in C^\infty([0, 1] \times M, N)$ such that $F(0, \cdot) = \tilde{u}$ and $F(1, \cdot) = \tilde{v}$, which implies that $\tilde{u}$ and $\tilde{v}$ are $W^{s,p}$ homotopic. Finally, u and v are $W^{s,p}$ homotopic. This shows the importance of the property P.

Theorem 5. *Each $u \in W^{s,p}(M, N)$ satisfies $P(u)$ when*

(a) *$sp \geq \dim M$,*
(b) *$0 < sp < 2, 0 < s < 1 + 1/p$,*
(c) *$\dim M = 2, 0 < s < 1 + 1/p$,*
(d) *$M = S^m, 0 < s < 1 + 1/p$,*