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Periodic solutions of first-order nonlinear functional differential equations ☆

X.H. Tang*, Zhiyuan Jiang

School of Mathematical Sciences and Computing Technology, Central South University, Changsha, Hunan 410083, PR China

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Abstract

In this paper, we obtain a set of "easily verifiable" sufficient conditions for the existence of the periodic solutions of the first-order linear functional differential equations with periodic perturbation

$$x'(t) = l(x)(t) + f(t, x_t),$$

where $l : C(\mathbf{R}) \to C(\mathbf{R})$ is a linear bounded operator. These conditions generalize and improve the known results given in the literature.

MSC: 34K13

Keywords: Functional differential equations; Periodic solution; Periodic perturbation

1. Introduction

Let $BC(\mathbf{R}, \mathbf{R})$ be the Banach space of bounded continuous functions $x : \mathbf{R} \to \mathbf{R}$ with the sup norm $\|x\| = \sup\{|x(t)| : t \in \mathbf{R}\}$ and let

$$C_\omega(\mathbf{R}) = \{x \in BC(\mathbf{R}, \mathbf{R}) : x(t + \omega) = x(t), t \in \mathbf{R}\},$$

where $\omega > 0$. Define the norms as follows:

$$\|x\|_0 = \max\{|x(t)| : 0 \le t \le \omega\}, \qquad \|x\|_1 = \int_0^\omega |x(t)|\mathrm{d}t, \quad \forall x \in C_\omega(\mathbf{R}).$$

Consider the first-order functional differential equations

$$x'(t) = l(x)(t) + f(t, x_t), \tag{1.1}$$

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* Corresponding author.
E-mail address: tangxh@mail.csu.edu.cn (X.H. Tang).

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and the particular case

$$x'(t) = l(x)(t) + g(t), \tag{1.2}$$

where $l : C(\mathbf{R}) \to C(\mathbf{R})$ is a linear bounded operator and $l(C_\omega(\mathbf{R})) \subseteq C_\omega(\mathbf{R})$, $g \in C_\omega(\mathbf{R})$, and $f \in C(\mathbf{R} \times BC(\mathbf{R}, \mathbf{R}), \mathbf{R})$, for any $t \in \mathbf{R}$, $f(t, \cdot)$ mapping a bounded set in $BC(\mathbf{R}, \mathbf{R})$ into a bounded set in $\mathbf{R}$ and satisfying $f(t+\omega, \varphi) = f(t, \varphi)$ for $(t, \varphi) \in \mathbf{R} \times BC(\mathbf{R}, \mathbf{R})$. Everywhere in what follows, we will assume that the operator l is nontrivial and admits the representation

$$l(x)(t) = -a(t)x(t) + l_1(x)(t) - l_2(x)(t), \tag{1.3}$$

where $a \in C_\omega(\mathbf{R})$, $l_1, l_2 : C_\omega(\mathbf{R}) \to C_\omega(\mathbf{R})$ are linear and satisfy the condition

$$l_1(x)(t) \geq 0, \qquad l_2(x)(t) \geq 0, \quad \forall t \in [0, \omega] \text{ if } x(t) \geq 0,\ \forall t \in [0, \omega].$$

In the representation (1.3), we list the instantaneous term $a(t)x(t)$ as an independent one, because the instantaneous term always plays an important role in the existence and stability of the periodic solution for Eq. (1.1).

The common particular case of Eq. (1.2) is the following linear equation with deviating arguments:

$$x'(t) = p_0(t)x(t) + \sum_{i=1}^{n} p_i(t)x(t - \tau_i(t)) + g(t), \tag{1.4}$$

where $g \in C_\omega(\mathbf{R})$ and $p_0, p_i, \tau_i \in C_\omega(\mathbf{R})$, $i = 1, 2, \ldots, n$. For Eq. (1.4), we find that

$$l_1(x)(t) = \sum_{i=1}^{n} [p_i(t)]_+ x(t - \tau_i(t)), \qquad l_2(x)(t) = \sum_{i=1}^{n} [p_i(t)]_- x(t - \tau_i(t)),$$

where here and in the sequel, $[x]_+ = (|x| + x)/2$, $[x]_- = (|x| - x)/2$.

It is known (see, e.g., [10]) that Eq. (1.2) has a unique ω-periodic solution if and only if the corresponding homogeneous equation

$$x'(t) = l(x)(t), \tag{1.5}$$

has only a trivial ω-periodic solution. In view of this fact, [1,10,16] gave a set of conditions which guarantee that Eq. (1.2) has a unique ω-periodic solution. In the paper [11], Ma, Yu and Wang further proved that the homogeneous equation (1.5) having only a trivial ω-periodic solution implies that Eq. (1.1) has at least one ω-periodic solution under the additional restriction

(H1) $\lim_{\|\varphi\| \to \infty} \frac{|f(t,\varphi)|}{\|\varphi\|} = 0$ uniformly in $t \in \mathbf{R}$.

In this paper, our main purpose is to derive a set of "easily verifiable" sufficient conditions for the existence of the periodic solutions of Eq. (1.1). These conditions generalize and improve the known results given in the literature (see [1,4,6,7,10,16]). For example, consider Eq. (1.4) with $p_0(t) \geq 0$ and $p_i(t) \leq 0,\ i = 1, 2, \ldots, n$. In paper [1], it is shown that if

$$\|p_0\|_1 < \frac{\sum_{i=1}^{n} \|p_i\|_1}{1 + \sum_{i=1}^{n} \|p_i\|_1} \tag{1.6}$$

and

$$\sum_{i=1}^{n} \|p_i\|_1 < 4\left(1 - \|p_0\|_1\right), \tag{1.7}$$

or

$$\sum_{i=1}^{n} \int_0^\omega |p_i(t)| \exp\left(\int_t^\omega p_0(s)\mathrm{d}s\right) < 4, \tag{1.8}$$