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Infinite periodic solutions to a class of second-order Sturm–Liouville neutral differential equations[☆]

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Abstract

By means of variational structure and Z_2-group index theory, we obtain infinite periodic solutions to a class second-order Sturm–Liouville neutral delay equations

$$(p(t)x'(t-s\tau))' - q(t)x(t-s\tau) + f(t, x(t), x(t-\tau), x(t-2\tau), \ldots, x(t-2s\tau)) = 0.$$

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1. Introduction

Recently, the existence and multiplicity of periodic solutions for second-order neutral differential equations have received a good deal of attention (see [2–7]).

But, for the existence of periodic solutions of functional differential equations, one commonly uses the method of fixed point theory, the coincidence degree theory, the Fourier analysis method etc; one rarely uses the means of critical point theory. In [8] and [9], the authors obtained multiple periodic solutions for a class of retarded differential equations by means of critical point theory and Z_p group index theory. Nevertheless, we note that these results were obtained by reducing retarded differential equations to related ordinary differential equations.

In [1], by using critical point theory and Z_2 group index theory, unlike the literature [8,9], we obtained the sufficient condition for there to exist infinite nontrivial $2\gamma\tau$-periodic solutions to the neutral differential equation (1.1) without reducing it to an ordinary differential equation:

$$x''(t-s\tau) + f(t, x(t), x(t-\tau), x(t-2\tau), \ldots, x(t-2s\tau)) = 0, \quad \tau > 0. \tag{1.1}$$

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In this paper, by using critical point theory and Z_2 group index theory, we obtain infinite periodic solutions to the second-order Sturm–Liouville neutral delay equations

$$(p(t)x'(t-s\tau))' - q(t)x(t-s\tau) + f(t, x(t), x(t-\tau), x(t-2\tau), \ldots, x(t-2s\tau)) = 0. \quad (1.2)$$

For the reader's convenience, we recall some basic definitions.

Definition 1.1. Let E be a real Banach space, and $f \in C^1(E, R)$. A critical point of f is a point where $f'(x) = 0$. A critical value of f is a number c such that $f(x) = c$ for some critical points x. K is a critical set where $K = \{x \in E \mid f'(x) = 0\}$, $K_c = \{x \in E \mid f'(x) = 0, f(x) = c\}$. f_c is a level set if $f_c = \{x \in E; \mid f(x) \leq c\}$.

Definition 1.2. Let E be a real Banach space, and $f \in C^1(E, R)$; we say that f satisfies the Palais–Smale condition if every sequence $\{x_n\} \subset E$ such that $\{f(x_n)\}$ is bounded and $f'(x_n) \to 0 (n \to \infty)$ has a converging subsequence.

Definition 1.3. Let E be a real Banach space, and $\Sigma = \{A \mid A \subset E \setminus \{\theta\}$ be a closed, symmetric set $\}$. Define $\gamma : \Sigma \to Z^+ \cup \{+\infty\}$ as follows:

$$\gamma(A) = \begin{cases} \min & \{n \in Z : \text{ there exists an odd continuous map } \varphi : A \to R^n \setminus \{\theta\}\}; \\ 0 & \text{If } A = \varnothing; \\ +\infty & \text{If there is no odd continuous map } \varphi : A \to R^n \setminus \{\theta\} \text{ for any } n \in Z. \end{cases}$$

We say that "γ is the genus of Σ ".

Lemma 1.4 (*[10] ChangKung Ching*). *Let $f \in C^1(X, R)$ be an even functional which satisfies the Palais–Smale condition and $f(\theta) = 0$. If*

(P_1) *there exist constants $\rho > 0$, $a > 0$ and a finite dimensional subspace E of X, such that $f(x)\,|_{E^{\perp} \cap S_\rho} \geq a$, where $S_\rho = \{x \in X : \|x\|_X = \rho\}$;*
(P_2) *for all finite dimensional subspaces $\widehat{E}$ of X, there is an $r = r(\widehat{E}) > 0$, such that $f(x) \leq 0$ for $x \in \widehat{E} \setminus B_r$.*

Then, f possesses an unbounded sequence of critical values.

Lemma 1.5. *Let E be a Hilbert space; if the weak convergence sequence $\{x_n\} \subset E$ (i.e., there exists x_0 such that $x_n \rightharpoonup x_0$) satisfies $\|x_n\| \to \|x_0\| (n \to \infty)$, then $\{x_n\}$ is convergent in E, i.e., $x_n \to x_0$.*

Proof. By

$$\begin{aligned} \|x_n - x_0\|^2 &= (x_n - x_0, x_n - x_0) \\ &= \|x_n\|^2 - (x_0, x_n) - (x_n, x_0) + \|x_0\| \quad (n = 1, 2, 3, \ldots) \end{aligned}$$

and continuity of the inner product, it is easy to see that

$$\lim_{n\to\infty} \|x_n - x_0\|^2 = \|x_0\|^2 - 2(x_0, x_0) + \|x_0\|^2 = 0,$$

that is to say $x_n \to x_0 (n \to \infty)$.

In this paper, we use Lemma 1.4 to deal with multiple periodic solutions of the system (1.2).

Our basic assumptions is that:

(A_1) $f(t, u_1, u_2, \ldots, u_{2s+1}) \in C(R^{2(s+1)}, R)$, and $\frac{\partial f(t,u_1,u_2,\ldots,u_{2s+1})}{\partial t} \neq 0$;
(A_2) there exists a continuously differentiable function $F(t, u_1, u_2, \ldots, u_s, u_{s+1}) \in C^1(R^{s+2}, R)$ such that

$$\begin{aligned} &F'_{u_{s+1}}(t, u_1, u_2, \ldots, u_s, u_{s+1}) + F'_{u_{s+1}}(t, u_2, u_3, \ldots, u_{s+2}) + \cdots + F'_{u_{s+1}}(t, u_{s+1}, u_{s+2}, \ldots, u_{2s+1}) \\ &\quad = f(t, u_1, u_2, \ldots, u_{2s+1}); \end{aligned}$$

(A_3) $F(t+\tau, u_1, u_2, \ldots, u_{s+1}) = F(t, u_1, u_2, \ldots, u_{s+1})$ for all $u_1, u_2, \ldots, u_s, u_{s+1} \in R$;
(A_4) $p(t), q(t) \in C^1[0, \tau]$ are τ-periodic functions and $p(t) > 0, q(t) > 0$;
(A_5) F satisfies: $F(t, -u_1, -u_2, \ldots, -u_{s+1}) = F(t, u_1, u_2, \ldots, u_{s+1})$, and

$$f(t, -u_1, -u_2, \ldots, -u_{2s+1}) = -f(t, u_1, u_2, \ldots, u_{2s+1}).$$