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Nonlinear versions of Stampacchia and Lax–Milgram theorems and applications to p-Laplace equations

Duong Minh Duc*, Nguyen Hoang Loc, Le Long Phi

Department of Mathematics and Computer Sciences, National University of Hochiminh City, 227 Nguyen Van Cu, Q5, Hochiminh City, Viet Nam

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Abstract

We obtain the nonlinear versions of the Stampacchia theorem and the Lax–Milgram theorem. Our results are stronger than the classical ones even in the linear case. Applying these theorems we get nontrivial solutions of p-Laplace elliptic and pseudo-p-Laplace problems.

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1. Introduction

To solve partial differential equations, we can use the following theorems (see [3,9,15]).

Theorem (*Lax Milgram*). *Let a be a continuous coercive bilinear form on a Hilbert space H and T be a continuous linear form on H. Then there is a unique vector u in H such that*

$$a(u, v) = T(v) \quad \forall v \in H.$$

Theorem (*Stampacchia*). *Let a be a continuous coercive bilinear form on a Hilbert space H, T be a continuous linear form on H, and K be a non-empty closed convex subset of H. Then there is a unique vector u in K such that*

$$a(u, u - v) \geq T(u - v) \quad \forall v \in K.$$

For nonlinear problems such as p-Laplace elliptic equations and inequalities, we can use the following extension of the Stampacchia theorem (see [3]).

Theorem (*Minty–Browder*). *Let E be a reflexive Banach space. Let A be a (nonlinear) continuous mapping from E into its dual space E' such that*

* Corresponding author.
E-mail address: dmduc@hcmc.netnam.vn (D.M. Duc).

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(i) $(Av - Aw)(v - w) > 0\ \forall v, w \in E, v \neq w$.
(ii) $\lim_{\|v\|\to\infty} \|v\|^{-1}(Av)(v) = \infty$.

Then, for any b in E' there is a unique u in E such that $Au = b$.

In applications to partial differential equations, $a(u, v)$, $T(v)$ and $Au(v)$ in the above theorems are usually $\mathrm{D}f(u)(v)$, $\mathrm{D}g(u)(v)$ and $\mathrm{D}J(u)(v)$, where $\mathrm{D}f$, $\mathrm{D}g$ and $\mathrm{D}J$ are derivatives of some functionals f, g and J respectively. This observation suggests relations between the above theorems and the Lagrange multiplier theorem. The essential difference between them is the presence of multipliers in the Lagrange multiplier theorem. In the linear case studied in [2] we used the homogeneity with different orders of two sides in the equation of the Lax–Milgram theorem to eliminate these multipliers and get the result. This technique could not be applied for nonlinear cases (see the left-hand side of the equation in the Minty–Browder theorem, for example).

In the present paper, we point out that we need not homogeneity but homogeneity inequalities to deal with inequalities in the Stampacchia theorem (see condition (FG) in Theorem 2.1). This technique cannot be applied to equations of the Lax–Milgram theorem. To overcome this difficulty, we prove that the Lagrange multiplier in this case should be 0 or 1 (see Theorem 2.2) and we get a generalized Lax–Milgram theorem.

In the second section of this paper we obtain nonlinear versions of these theorems by applying the Lagrange multiplier theorem in [2]. Our results are stronger than the classical ones even in the linear case.

Applying these theorems we get nontrivial solutions of p-Laplace and pseudo-elliptic equations and inequalities in the last sections. Our results can be applied to equations to which the methods in [4–8,10–14] are not applicable. When one uses the mountain-pass theorem, the boundedness of the Palais–Smale sequence requires some conditions of functionals that we do not need here in applying the techniques of the Lagrange multiplier theorem.

2. Nonlinear versions of the Stampacchia theorem and the Lax–Milgram theorem

First we need some definitions and notations.

Definition 2.1. Let φ be a mapping from a normed space $(X, \|.\|_X)$ into $[-\infty, \infty]$. We say φ is of class $\mathcal{A}(X)$ if there are two mappings η and μ from $[0, \infty) \times [0, \infty)$ and $[0, \infty)$ respectively into $[0, \infty)$ such that

(A1) $|\varphi(x + y)| \leq \eta(|\varphi(x)|, |\varphi(y)|)\ \forall x, y \in X$,
(A2) $|\varphi(tx)| \leq \mu(|t|)|\varphi(x)|\ \forall t \in \mathbb{R}, x \in X$.

For any given mapping φ of class $\mathcal{A}(X)$ we put

$$V_{X,\varphi} = \{x \in X : |\varphi(x)| < \infty\}.$$

It is clear that $V_{X,\varphi}$ is a linear subspace of X for any φ in $\mathcal{A}(X)$.

Definition 2.2. Let U be an open subset of a normed space $(X, \|.\|_X)$, G be a linear subspace of X, and f be a function from U into $\mathbb{R}$. For any $(x, h) \in U \times G$ we define

$$U(x, h) = \{t \in \mathbb{R} : x + th \in U\} \quad \text{and}$$
$$f_{(x,h)}(t) = f(x + th) \quad \forall t \in U(x, h).$$

Since U is open in X, it is clear that $U(x, h)$ is an open subset of $\mathbb{R}$. We say
(i) f is G-continuous on U if and only if $f_{(x,h)}$ is continuous at 0 for any (x, h) in $U \times G$.
(ii) f is G-differentiable at x if and only if there exists a linear mapping $\mathrm{D}f(x)$ from G into $\mathbb{R}$ such that

$$\lim_{t\to 0} \frac{f(x + th) - f(x)}{t} = \mathrm{D}f(x)(h) \quad \forall h \in G.$$

(iii) f is G-differentiable on U if f is G-differentiable at every x in U.
(iv) f is strongly G-differentiable at x if and only if there exists a linear mapping $\mathrm{D}f(x)$ from G into $\mathbb{R}$ such that

$$\lim_{(s,t)\to 0} \frac{f(x + sth + sk) - f(x)}{s} = \mathrm{D}f(x)(k) \quad \forall h, k \in G.$$