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Strongly resonant quasilinear elliptic equations

Leszek Gasiński*

Jagiellonian University, Institute of Computer Science, ul. Nawojki 11, 30072 Kraków, Poland

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Abstract

In this paper we study the following nonlinear boundary value problem

$$\begin{cases} -\Delta_p u = \lambda_1 |u|^{p-2} u + g(u) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

An existence result is shown under some strong resonance conditions generalizing those of Tang and Landesman–Lazer.

MSC: 34B15; 35J25; 47J30; 49J40

Keywords: Quasilinear elliptic equation; Strong resonance; Cerami condition; Saddle point theorem; Tang condition

1. Introduction

In this paper we study the existence of a solution for the following nonlinear boundary value problem

$$\begin{cases} -\Delta_p u = \lambda_1 |u|^{p-2} u + g(u) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where $1 < p < +\infty$, Ω is a bounded and open set in $\mathbb{R}^N$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function. By $\Delta_p u$ we denote the p-Laplacian of u, i.e. $\Delta_p u = \operatorname{div}(|\nabla u|^{p-1} \nabla u)$. This problem was studied in more general form:

$$\begin{cases} -\Delta_p u = \lambda_1 |u|^{p-2} u + g(u) - h(x) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.2}$$

by Tang [10] (with $N = 1$ and $p = 2$ — the semilinear case), Bouchala [3,4] (with $N = 1$) and Bouchala and Drábek [5] (general case).

In this paper we prove the existence of a solution to problem (1.1) by imposing some generalization of the Tang condition (cf. Tang [10]) so also those of Landesman and Lazer [7]. The method of the proof is motivated by the papers of Arcoya and Orsina [2] and Bouchala and Drábek [5].

* Tel.: +48 501363981; fax: +48 126341865.
E-mail address: leszgasi@virgo.ii.uj.edu.pl.

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First let us state our assumptions containing a generalization of the Tang condition. Together with the function g we consider its primitive

$$G(\zeta) = \int_0^\zeta g(s)\mathrm{d}s.$$

Let $\eta\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a function satisfying the following hypotheses:
$\mathrm{H}(\eta)$ $\eta\colon \mathbb{R} \longrightarrow \mathbb{R}$ is a function such that

(i) $\zeta\eta(\zeta) > 0$ for all $\zeta \in \mathbb{R} \setminus \{0\}$;
(ii) $\eta(\zeta) \longrightarrow \pm\infty$ as $\zeta \to \pm\infty$.

For any such function η, we define the function

$$F_\eta(\zeta) = \begin{cases} \dfrac{pG(\zeta) - \zeta g(\zeta)}{\eta(\zeta)} & \text{if } \zeta \neq 0, \\ (p-1)g(0) & \text{if } \zeta = 0. \end{cases}$$

If η is continuous, then F_η is also continuous in $\mathbb{R} \setminus \{0\}$. Its behaviour at $\zeta = 0$ depends on η. We also set

$$F_\eta|_{-\infty} = \liminf_{\zeta\to-\infty} F_\eta(\zeta), \qquad F_\eta|^{-\infty} = \limsup_{\zeta\to-\infty} F_\eta(\zeta),$$
$$F_\eta|_{+\infty} = \liminf_{\zeta\to+\infty} F_\eta(\zeta), \qquad F_\eta|^{+\infty} = \limsup_{\zeta\to+\infty} F_\eta(\zeta).$$

As for the function g we will consider one of the following two sets of assumptions:
$\mathrm{H(g)}_1$

(i) $g\colon \mathbb{R} \longrightarrow \mathbb{R}$ is continuous;
(ii) $\lim_{|\zeta|\to+\infty} \frac{g(\zeta)}{|\zeta|^{p-1}} = 0$
(iii) $F_\eta|^{-\infty} < 0 < F_\eta|_{+\infty}$.

$\mathrm{H(g)}_2$

(i) $g\colon \mathbb{R} \longrightarrow \mathbb{R}$ is continuous;
(ii) $\lim_{|\zeta|\to+\infty} \frac{g(\zeta)}{|\zeta|^{p-1}} = 0$
(iii) $F_\eta|_{-\infty} > 0 > F_\eta|^{+\infty}$.

Remark 1.1. First note that if hypothesis $\mathrm{H(g)}_1$(iii) (respectively $\mathrm{H(g)}_2$(iii)) holds for some η satisfying hypotheses $\mathrm{H}(\eta)$, then it also holds for any $\widetilde{\eta}$ satisfying hypotheses $\mathrm{H}(\eta)$ and such that

$$|\widetilde{\eta}(\zeta)| \leqslant |\eta(\zeta)| \quad \forall \zeta \geqslant \zeta_0,$$

with some $\zeta_0 \geqslant 0$. In particular, if the hypothesis holds for the function $\eta(\zeta) = \zeta$, then it also holds for any function $\widetilde{\eta}(\zeta) = \zeta|\zeta|^{\mu-1}$, with any $\mu \in (0, 1]$. Thus in the case when $h \equiv 0$ in (1.2), our assumptions are more general than those of Tang [10], Bouchala [3,4], Bouchala and Drábek [5].

Remark 1.2. Let us consider the following function:

$$g(\zeta) = \begin{cases} \dfrac{1}{\zeta} - \dfrac{\ln(1+\zeta^2)}{\mathrm{e}^{\zeta^4|\sin\zeta|}} & \text{for } \zeta \geqslant \pi, \\ a_0\zeta + 1 & \text{for } 0 \leqslant \zeta \leqslant \pi, \\ 2\mathrm{e}^\zeta - 1 & \text{for } \zeta \leqslant 0 \end{cases}$$

with $a_0 = \frac{1-\pi}{\pi^2} - \frac{\ln(1+\pi^2)}{\pi}$ (the "middle" term defined on the interval $[0, \pi]$ is just the linear function making the function g continuous and it is not important in our considerations). One can check that taking

$$\eta(\zeta) = \begin{cases} \ln\zeta & \text{for } \zeta \geqslant \mathrm{e}, \\ \zeta & \text{for } \zeta < \mathrm{e}, \end{cases}$$