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A Liouville-type theorem and the decay of radial solutions of a semilinear heat equation

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Abstract

We consider the semilinear parabolic equation $u_t = \Delta u + u^p$ on $\mathbb{R}^N$, where the power nonlinearity is subcritical. We first address the question of existence of entire solutions, that is, solutions defined for all $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$. Our main result asserts that there are no positive radially symmetric bounded entire solutions. Then we consider radial solutions of the Cauchy problem. We show that if such a solution is global, that is, defined for all $t \geqslant 0$, then it necessarily converges to 0, as $t \to \infty$, uniformly with respect to $x \in \mathbb{R}^N$.

MSC: 35k15; 35b40

Keywords: Subcritical semilinear heat equation; Entire solutions; Liouville theorem; Decay of global solutions

1. Introduction

In this paper we study nonnegative solutions of parabolic equations of the form

$$u_t = \Delta u + u^p, \quad x \in \mathbb{R}^N, \ \ t \in \mathbb{R}, \tag{1.1}$$

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where $p > 1$. We are particularly interested in classical solutions defined for all $t \in \mathbb{R}$, that is, solutions contained in $C^{2,1}(\mathbb{R}^N \times \mathbb{R})$. Below we refer to such solutions as *entire solutions*. We address the question of existence of positive entire solutions. In case $p \geqslant p_S$, where p_S is the Sobolev critical exponent,

$$p_S := \begin{cases} \dfrac{N+2}{N-2} & \text{if } N \geqslant 3, \\ \infty & \text{if } N \in \{1, 2\}, \end{cases} \tag{1.2}$$

such solutions exist. In fact, there are positive bounded steady states, or time-independent solutions. On the other hand, it has been known for some time that no positive bounded steady states exist for $p < p_S$ (see [8,4]). The question of existence of more general entire positive solutions has not been settled yet in the entire subcritical range. The following partial results are available. For $p \leqslant 1 + 2/N$, classical Fujita-type results rule out positive bounded solutions even on $(0, \infty)$. A result of Bidaut-Véron [2] implies nonexistence of positive bounded entire solutions in the larger range $p < N(N+2)/(N-1)^2$. As consequence of a result of Merle and Zaag [12], one obtains that for any $p < p_S$ there are no positive entire solutions satisfying the extra condition

$$\limsup_{t \to -\infty} |t|^{1/(p-1)} \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} < \infty. \tag{1.3}$$

More specifically, they proved that any such solution is necessarily constant in space, thus, being a positive solution of $u_t = u^p$, it blows up in finite time. For dimensions $N \leqslant 3$ (and any $p < p_S$), the nonexistence of positive entire solutions which are radial and radially decreasing has been proved by Matos and Souplet [11]. In this paper we prove the nonexistence in the radial case without any monotonicity assumption and without restrictions on the dimension.

Theorem 1.1. *Let $p < p_S$. If $u = u(|x|, t)$ is a nonnegative radial entire solution of* (1.1) *such that u is bounded in $\mathbb{R}^N \times (-\infty, T)$ for some $T \in \mathbb{R}$ then $u \equiv 0$.*

Our proof is based on intersection comparison arguments (therefore the restriction to the radial case) and on properties of (sign changing) steady states of (1.1). Whenever we consider sign changing solutions the nonlinearity u^p is interpreted as $|u|^{p-1}u$.

Liouville-type results for elliptic and parabolic equations have proved very useful in many applications. For example, combined with scaling arguments they yield a priori bounds on positive steady states or time-dependent solutions of the problem

$$u_t - \Delta u = \lambda u + a(x)u^p, \quad x \in \Omega, \quad t > 0,$$

$$u = 0, \quad x \in \partial\Omega, \quad t > 0,$$

$$u(x, 0) = u_0(x) \geqslant 0, \quad x \in \Omega,$$

on bounded domains Ω. Such bounds play a crucial role in the study of equilibria, heteroclinic orbits and blow-up. See [6,11,13,14] and the references therein for a discussion of these and more general results.