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# Energy decay rates for the semilinear wave equation with nonlinear localized damping and source terms

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**Abstract**

In this paper we develop an intrinsic approach to derivation of energy decay rates for the semilinear wave equation with *localized* interior *nonlinear* monotone damping $g(u_t)$ and a *source term* $f(u)$. The proposed approach allows to consider, in an unified way, much more general classes of hyperbolic problems than addressed before in the literature. These generalizations refer to both geometric and topological aspects of the problem.

The method leads to optimal decay rates for solutions of semilinear hyperbolic equations driven by a source of critical exponent and subjected to a nonlinear damping localized in a small region adjacent to a *portion* of the boundary. The distinct features of the model include: (i) *Neumann* boundary conditions are assumed and, (ii) no growth conditions are imposed on the damping $g(s)$. It is well known that Neumann boundary does not satisfy Lopatinski condition and, therefore, the analysis of propagation of energy in the absence of the damping on the Neumann part of the boundary requires special geometric considerations. In addition, the sole conditions assumed on $g(s)$ are monotonicity, continuity and $g(0)=0$. In particular, no differentiability and no growth conditions are imposed on the damping both at the origin and at the infinity. The asymptotic decay rates for the energy function are obtained from an intrinsic algorithm driven by solutions of simple ODE. Several examples illustrate

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the theory by exhibiting various decay rates (exponential, algebraic, rational, logarithmic, etc.) for the energy functional.

An important corollary of our energy decay theorem is a stability result which shows that, under certain conditions, when dissipation is sublinear at infinity, the solution of the system remains uniformly bounded for all time in the norms above the finite energy level, even in the presence of a nonlinear source term.


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## 1. Introduction

Let $\Omega$ be an open bounded connected domain in $\mathbb{R}^n$, with locally Lipschitz boundary $\Gamma$. Define $Q_T \equiv [0, T] \times \Omega$, $\Sigma_T \equiv [0, T] \times \Gamma$, and let $\|\cdot\|$ stand for $L^2(\Omega)$ norm.

Consider the following model of the wave equation with localized damping $\chi g(u_t)$ and source term $f(u)$:

$$\begin{cases} u_{tt} - \Delta u + \chi g(u_t) = f(u) \quad \text{in } Q_T, \\ u(0) = u_0, \\ u_t(0) = u_1. \end{cases} \tag{1}$$

The functions $g$ (resp. $f$) represent Nemytski operators associated with scalar, continuous real-valued functions $g(s)$ (resp. $f(s)$). Function $g(s)$, assumed monotone increasing, models dissipation in the equation. Instead, function $f(s)$ corresponds to the modeling of a source. The dissipation is assumed to act on small subportion of the domain $\Omega$, hence we introduce a map $\chi$ which is the characteristic function of a subset $\Omega_\chi$ of $\Omega$. Precise description of $\Omega_\chi$ will be given later, for now it suffices to say that $\Omega_\chi$ covers a thin layer (a collar) near a portion of the boundary.

The aim of this paper is to study asymptotic behavior (as $t \to \infty$) and related decay rates for the corresponding solutions evolving in the standard for the wave equation *finite energy* space $H^1(\Omega) \times L^2(\Omega)$ (precise definition of the energy space will be given later). In order to analyze the problem and state the results, we must specify boundary conditions (BC) for (1). As we shall see below, BC play an important role in our analysis; we are predominantly interested in the Neumann-type BC, which do not satisfy Lopatinski condition and therefore are the most challenging in the context of this problem.

### 1.1. Boundary conditions

A distinct feature of the paper is the analysis of dynamics under *Neumann* BC, that do not satisfy *Lopatinski* condition. Before specifying the boundary dynamics, we introduce a few definitions. Divide the boundary $\Gamma$ into parts: $\Gamma_0$ and $\Gamma_1$ so that $\Gamma = \overline{\Gamma_0} \cup \overline{\Gamma_1}$. We set