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Instability of solitary waves for a generalized Benney–Luke equation☆

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Abstract

We study linear instability of solitary wave solutions of a one-dimensional generalized Benney–Luke equation, which is a formally valid approximation for describing two-way water wave propagation in the presence of surface tension. Further, we implement a finite difference numerical scheme which combines an explicit predictor and an implicit corrector step to compute solutions of the model equation which is used to validate the theory presented.

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Keywords: Boussinesq equation; Solitary wave; Linear instability; Eigenvalue problem

1. Introduction

In this paper we shall study linear instability of solitary waves of the nonlinear Benney–Luke equation

$$\Phi_{tt} - \Phi_{xx} + a\Phi_{xxxx} - b\Phi_{xxtt} + n\Phi_t(\Phi_x)^{n-1}\Phi_{xx} + 2(\Phi_x)^n\Phi_{xt} = 0, \tag{1}$$

where $\Phi = \Phi(x,t)$ for $x, t \in \mathbb{R}$, a and b are positive numbers such that $a - b = \sigma - 1/3$. The dimensionless parameter σ is named the Bond number, which captures the effects of surface tension and gravity force.

In contrast with water wave models which are formally valid only for one-way propagating waves, such as the generalized Korteweg–de Vries equations (GKdV) and other long wave approximations such as the regularized long wave equation and regularized Boussinesq equation (see for example Eq. (1.1.B1) in Pego and Weinstein [17]), the Benney–Luke equation (1) has the advantage of being a formally valid approximation for describing two-way water wave propagation, which includes the effect of surface tension controlled by the parameters a and b. In particular, we note that generalized Benney–Luke equation (1) is the one-dimensional version (after rescaling) of the generalized Benney–Luke equation

$$\Phi_{tt} - \Delta\Phi + \mu(a\Delta^2\Phi - b\Delta\Phi_{tt}) + \epsilon\left(\Phi_t\Delta_n\Phi + \left(\frac{2}{n+1}\right)|\nabla^{\frac{n+1}{2}}\Phi|_t^2\right) = 0, \tag{2}$$

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where

$$\left|\nabla^r \phi\right| = \sqrt{|\partial_x \phi|^{2r} + \left|\partial_y \phi\right|^{2r}} \quad \text{and} \quad \Delta_n \phi = \nabla \cdot \left(\nabla^n \phi\right) = \partial_x \left[\partial_x \phi\right]^n + \partial_y \left[\partial_y \phi\right]^n.$$

Existence and analyticity of solitary waves for the generalized Benney–Luke equation were established by Quintero in [19].

An important observation regarding this model, which will be crucial in our analysis of instability, is that in the one-dimensional case, the unscaled Eq. (2) reduces formally in a suitable limit to the (GKdV) equation;

$$\eta_\tau - \left(\sigma - \frac{1}{3}\right)\eta_{XXX} + (n+2)\eta^n \eta_X = 0, \tag{3}$$

where η represents the elevation.

Among a variety of stability results for diverse problems (see works by Benjamin, Bona, Alexander, Jones, Shatah, Weinstein, Pego, Smereka, Souganidis, Strauss, among others [1–4,15–17,23]) there exists an important result due to Grillakis, Shatah, and Strauss (see [8]) for studying orbital stability of travelling wave solutions for problems with a general Hamiltonian structure. In this case, travelling waves ϕ_c of wave speed $c > 0$ are characterized as critical points of some energy functional $\mathcal{F}$ and the stability result follows by analyzing the second variation of $\mathcal{F}(\phi_c)$ about solitary waves.

It is important to point out that the one-dimensional Benney–Luke equation (1) does not fit into the class of abstract Hamiltonian systems studied by Grillakis et al., when $c^2 > \max\{1, a/b\}$ and $n \geq 1$, preventing us from using it to establish nonlinear stability of solitary waves. However, we now know that in a work in preparation, Pego and Quintero [18] are considering the analysis of the linear stability and the asymptotic stability of travelling waves for Eq. (1), when the wave speed satisfies $c^2 > \max\{1, a/b\}$ and $n = 1$. On the other hand, in recent works, J. Quintero established nonlinear orbital stability of solitary waves for the 1-D and 2-D Benney–Luke equation (1), when the wave speed satisfies the complementary range: $n = 1$, $c > 0$ and $c^2 < \min\{1, a/b\}$. The 1-D case follows as a consequence of Grillakis et al. work, but in the 2-D case the verification of orbital stability is obtained directly without using the Grillakis et al. work (see [20,21]).

Despite the difficulty of getting nonlinear stability (or instability), there have been some attempts to obtain linear stability of solitary waves for generalized Korteweg–de Vries equations (GKdV) or Boussinesq type equations, as convective linear stability, by using an appropriate exponentially weighted L^2 space:

$$L^2_\alpha = \left\{ f : \mathbb{R} \to \mathbb{R} : \mathrm{e}^{\alpha s} f \in L^2 \right\} \tag{4}$$

with norm

$$\|f\|_\alpha = \|\mathrm{e}^{\alpha s} f\|_{L^2} = \left(\int_{\mathbb{R}} \mathrm{e}^{2\alpha s} |f(s)|^2 \,\mathrm{d}s\right)^{\frac{1}{2}}, \quad (\alpha \text{ small enough}) \tag{5}$$

and also by observing that in a frame of reference moving with the speed of the unperturbed solitary wave, the linearized equation for the solitary wave perturbation $z(s, \tau)$ is written in the form

$$\partial_\tau z = \mathcal{B} z. \tag{6}$$

For such problems, there exists a two-parameter family of non-decaying solutions, corresponding to infinitesimal shifts in phase and changes in solitary wave speed. In some cases, it has been shown that Eq. (6) is asymptotically stable with respect to the weighted norm in (5), modulo this two-parameter family (see [17]).

The main issue is that the effect of using the weighted space L^2_α is to shift the continuous spectrum of $\mathcal{B}$ from the imaginary axis into the left half-plane. For some problems, there are no nonzero eigenvalues satisfying $\Re(\lambda) > -\rho$ for some $\rho > 0$, with respect to the weighted norm, and some sort of linear stability can be established. We want to note that these linear stability results for solitary waves for Boussinesq equations have been obtained by using a technique developed by Evans (see [5]) introduced to study stability of impulses for the nerve axon equation, (see [1, 9,15–17,10]). In those cases, eigenvalues in the right half-plane of the linearization of a Boussinesq equation about a solitary wave are characterized as zeros of an analytic function, called the Evans function. The difficulty with this approach is that in general Evans functions are unknown. However, this difficulty can be minimized by using a strong