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Locally adjoint mappings and optimization of the first boundary value problem for hyperbolic type discrete and differential inclusions

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Abstract

The present paper deals with discrete approximation on a uniform grid of the first boundary value problem (P_C) for differential inclusions of hyperbolic type. In the form of Euler–Lagrange inclusions, necessary and sufficient conditions for optimality are derived for the discrete (P_D) and continuous (P_C) problems on the basis of new concepts of locally adjoint mappings. The results obtained are generalized to the multidimensional case with a second order elliptic operator.

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1. Introduction

The present work is devoted to an investigation of problems described by so-called discrete and differential inclusions of hyperbolic type. It is known that extremal problems concerning multivalued mappings with lumped and distributed parameters constitute one of the intensively developable fields of optimal control theory [1–9,14,16,26,28,33–38]. In [5,21] necessary and sufficient conditions for hyperbolic differential inclusions of Goursat–Darboux type are given. The papers [4,6,7,29] are dedicated to the existence of solutions and other qualitative problems of hyperbolic differential inclusions.

A lot of problems in economic dynamics, as well as classical problems on optimal control in vibrations, chemical, hydrodynamical engineering, heat, diffusion processes, differential games, and so on, can be reduced to such investigations. We refer the reader to the survey papers [1–9,11–13,22,26,30,34,36–38]. Now let us explain the principal method that we use to obtain the mentioned results. The paper is organized as follows.

In Section 2 first there are given some suitable definitions and supplementary notions that constitute a certain method which facilitates obtaining necessary and sufficient conditions. Then the first initial boundary value problems are formulated for so-called hyperbolic discrete (P_D) and differential inclusions (P_C), (P_M).

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In Section 3 for problem (P_D) we use one of the constructions of convex and nonsmooth analysis to get necessary and sufficient conditions for optimality. The latter can be reduced to finite dimensional problems of mathematical programming, namely to minimization of functions on the intersection of the finite number of sets. In the reviewed results the arising adjoint inclusions are stated in the Euler–Lagrange form [16,28]. Note that because of the new construction of locally adjoint mappings (LAM) this form automatically implies the Weierstrass–Pontryagin maximum condition. Another definition of the LAM is introduced by Pshenichnyi [30] and successively applied in the papers [17,19–21] of Mahmudov for ordinary and different partial differential inclusions. Besides, a similar concept is given by Mordukhovich [28] too and is called the coderivative of multifunctions at a point. Moreover it appears that the use of the convex upper approximations (CUA) for nonconvex functions and local tents [30] are very suitable for obtaining the optimality conditions for the posed problems. Observe that the main successful application of local approximations and the transition to convex approximations of sets is the establishment of necessary conditions for nonconvex optimization problems. In the field of different convex and nonconvex approximations of functions and sets the reader can also consult Clarke et al. [9], Demianov and Vasilev [10], Frankowska [13], Mordukhovich [25,28], Pshenichnyi [30], Rockafellar [27,31] for related and additional material.

In Section 4 we develop the method of discrete approximations to obtain the necessary and sufficient conditions for optimality for the discrete approximation problem. We obtain discrete analogues of the Euler–Lagrange conditions for the first boundary value problem. With this aim we use difference approximations of partial derivatives and grid functions on a uniform grid to approximate the first boundary value problem for differential inclusions of hyperbolic type. The latter is possible by passing to necessary conditions for an extremum of discrete hyperbolic inclusions (P_D) in Section 3. It is found that the method concerned requires some special equivalence theorems of a LAM, which arises in discrete and discrete approximation problems. The equivalence theorems obtained allow us to make the transition between problems (P_D) and (P_C). Obviously, such difference problems, in addition to being of independent interest, can play an important role also in computational procedures.

In Section 5 we describe the tools of the locally adjoint mappings for convex and nonconvex cases used in the paper to obtain sufficient optimality conditions for hyperbolic differential inclusions. The derivation of sufficient conditions is implemented by passing to the formal limit as the discrete steps tend to zero. Of course, by using the suggested methods for ordinary differential inclusions of Mordukhovich [27,28] or Pshenichnyi [30] it can be proved that the sufficient conditions obtained are also necessary for optimality. At the end of Section 5 we consider a linear optimal control problem of hyperbolic type. This example shows that in known problems the adjoint inclusion coincides with the adjoint equation which is traditionally obtained with the help of the Hamiltonian function.

Some duality relations and optimality conditions for an extremum of different control problems with partial differential inclusions can be found in [2,4,5,9,17–20].

Note that in the considered hyperbolic differential inclusions the solution is taken in the space of classical solutions. In Section 6 the results obtained are generalized to the multidimensional case with a second order elliptic operator (P_M). Therefore, at the end of the paper we indicate general ways of extending the results to the case of generalized solutions [24].

2. Necessary concepts and problem statements

Let R^n be the n-dimensional Euclidean space; (u_1, u_2) is a pair of elements $u_1, u_2 \in R^n$ and $\langle u_1, u_2\rangle$ is their inner product. A multivalued mapping $F : R^{4n} \to 2^{R^n}$ is convex if its graph $\mathrm{gph}F = \{(u_1, u_2, u_3, u_4, v) : v \in F(u_1, u_2, u_3, u_4)\}$ is a convex subset of R^{5n}. It is convex valued if $F(u)$ is a convex set for each $u = (u_1, u_2, u_3, u_4) \in \mathrm{dom}\, F = \{u : F(u) \neq \emptyset\}$. F is closed if $\mathrm{gph}F$ is a closed set in R^{5n}.

Let us introduce the notation

$$M(u, v^*) = \sup_v \{\langle v, v^*\rangle : v \in F(u)\}, \quad v^* \in R^n,$$

$$F(u, v^*) = \{v \in F(u) : \langle v, v^*\rangle = M(u, v^*)\}.$$

For convex F we let $M(u, v^*) = -\infty$ if $F(u) = \emptyset$.

Let ri A be the relative interior of a set $A \subset R^n$, i.e., the set of interior points of A with respect to its affine hull Aff$A$. Note that a convex cone $K_A(u_0)$ at a point $u_0 \in A$ is a cone of tangent directions if from inclusion $\bar{u} \in K_A(u_0)$, it follows that $\bar{u}$ is a tangent vector at a given point $u_0 \in A$ [30]. Obviously, at a fixed point of an arbitrary set A, there