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Pseudo-almost periodicity of some nonautonomous evolution equations with delay☆

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Abstract

This paper is concerned with pseudo-almost periodicity of the solutions to the nonautonomous evolution equation with delay $u'(t) = A(t)u(t) + f(t, u(t-h))$. Some sufficient conditions which ensure the existence and uniqueness of pseudo-almost periodic mild solutions to the evolution equation with delay are given. An example is shown to illustrate our results.

Keywords: Pseudo-almost periodic; Almost periodic; Evolution family; Exponential dichotomy

1. Preliminaries

In this paper, we investigate the pseudo-almost periodicity of the solutions to the following nonautonomous evolution equation with delay:

$$u'(t) = A(t)u(t) + f(t, u(t-h)), \quad t \in \mathbb{R} \tag{1.1}$$

in a Banach space X, where $h \geq 0$ is a fixed constant, and $A(t)$ and $f(t, u)$ satisfy the hypotheses (H1)–(H4) recalled in Section 2.

Recently, the existence of pseudo-almost periodic solutions to various differential equations has been of great interest for many researchers (cf. [3,8–10,16,21] and references therein). Many authors have studied the pseudo-almost periodicity of the solutions to Eq. (1.1) in the case where $A(t) = A$ and $h = 0$ (see, e.g., [3,8,10,16]). More precisely, in [3,8] the existence and uniqueness of pseudo-almost periodic solutions to some semilinear differential equations has been considered in the case where A is a so-called Hille–Yosida operator. In [16], such a problem has been studied when A is the infinitesimal generator of a compact semigroup. The problem has also been investigated in the case of $-A$ generating an analytic semigroup in [10].

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However, there exists little work concerning the pseudo-almost periodicity of solutions to (1.1) in Banach spaces and in the nonautonomous case. In this paper, we will show that (1.1) has a unique pseudo-almost periodic mild solution under some suitable assumptions on $A(t)$ and f. We notice that Eq. (1.1) is an important model for practical problems, and its various forms have been studied in many papers (see, e.g., [4,5,12] and references therein).

Recall that the concept of pseudo-almost periodicity is a natural generalization of the classical concept of almost periodicity in the sense of Bochner and this new concept is welcome for implementing another interesting generalization of almost periodicity, the so-called asymptotical almost periodicity due to Fréchet (cf. [13,18]). For more details on the concepts of almost periodicity and pseudo-almost periodicity, we refer the readers to [3,8–10,13, 15,16,18].

Throughout this paper, we denote by $C_b(\mathbb{R}, X)$ the Banach space of bounded continuous functions from $\mathbb{R}$ to X with supremum norm. Similarly, $C_b(\mathbb{R} \times X, X)$ is the Banach space of bounded continuous functions from $\mathbb{R} \times X$ to X with supremum norm. For the reader's convenience, we recall some definitions of almost periodicity and pseudo-almost periodicity.

Definition 1.1. $f \in C_b(\mathbb{R}, X)$ is called almost periodic if for each $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that every interval I of length $l(\varepsilon)$ contains a number τ with the property that

$$\|f(t+\tau) - f(t)\| < \varepsilon \quad \text{for all } t \in \mathbb{R}.$$

We denote by $AP(X)$ the set of all such functions.

Definition 1.2. $f \in C_b(\mathbb{R} \times X, X)$ is called almost periodic in t uniformly for $x \in X$ if for each $\varepsilon > 0$ and for each compact subset E of X there exists $l(\varepsilon) > 0$ such that every interval I of length $l(\varepsilon)$ contains a number τ with the property that

$$\|f(t+\tau, x) - f(t, x)\| < \varepsilon \quad \text{for all } t \in \mathbb{R},\ x \in E.$$

We denote by $AP(\mathbb{R} \times X, X)$ the set of all such functions.

Set

$$AP_0(X) = \left\{\varphi \in C_b(\mathbb{R}, X) : \lim_{r \to +\infty} \frac{1}{2r} \int_{-r}^{r} \|\varphi(t)\| \mathrm{d}t = 0\right\}.$$

Denote by $AP_0(\mathbb{R} \times X, X)$ the space of all functions $\varphi \in C_b(\mathbb{R} \times X, X)$ such that

$$\lim_{r \to +\infty} \frac{1}{2r} \int_{-r}^{r} \|\varphi(t, x)\| \mathrm{d}t = 0$$

uniformly in $x \in X$.

Definition 1.3. $f \in C_b(\mathbb{R}, X)(C_b(\mathbb{R} \times X, X))$ is called pseudo-almost periodic if

$$f = g + \varphi$$

with $g \in AP(X)(AP(\mathbb{R} \times X, X))$ and $\varphi \in AP_0(X)(AP_0(\mathbb{R} \times X, X))$.

Denote by $PAP(X)(PAP(\mathbb{R} \times X, X))$ the set of all such functions. We know that $PAP(X)$ is a closed subspace of $C_b(\mathbb{R}, X)$ from [16, Lemma 1.2]. So $PAP(X)$ is a Banach space.

We also need to recall some notation concerning exponential dichotomy. An evolution family U is called hyperbolic (or has exponential dichotomy) if there are projections $P(t)$, $t \in \mathbb{R}$, uniformly bounded and strongly continuous in t, and constants $M, \omega > 0$ such that

(a) $U(t,s)P(s) = P(t)U(t,s)$ for all $t \geq s$,
(b) the restriction $U_Q(t,s) : Q(s)X \to Q(t)X$ is invertible for all $t \geq s$ (and we set $U_Q(s,t) = U_Q(t,s)^{-1}$),
(c) $\|U(t,s)P(s)\| \leq M\mathrm{e}^{-\omega(t-s)}$ and $\|U_Q(s,t)Q(t)\| \leq M\mathrm{e}^{-\omega(t-s)}$ for all $t \geq s$.