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Nonlinear Analysis 68 (2008) 1825–1843
Nonlinear Analysis
www.elsevier.com/locate/na

Asymptotic behavior of the solution to the non-isothermal phase separation

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Received 16 November 2006; accepted 9 January 2007

Abstract

We consider the non-isothermal phase separation models of the Penrose–Fife type, which were proposed in [O. Penrose, P.C. Fife, Thermodynamically consistent models of phase-field type for the kinetics of phase transitions, Physica D 43 (1990) 44–62], with homogeneous Neumann boundary conditions on the nonlinear heat flux $\mathbf{q} = \nabla\alpha(u)$, i.e., $\mathbf{q}\cdot\mathbf{n} = 0$ on the boundary of a region which the material occupies. Here u represents the absolute temperature. For this model, we first show that there exists a unique solution globally in time.

Moreover, the ω-limit set associated with the trajectory of the unique global solution is non-empty, connected, and compact in some suitable space; as well as being composed of solutions to the steady state problem. For the stability of stationary solutions, we show that the dynamically stable solutions to the steady state problem are characterized by linearized stable solutions to the elliptic problem with a non-local term, which is equivalent to our steady state problem.

MSC: 35K45; 35K55; 37L15

Keywords: Non-isothermal phase separation; Penrose–Fife model; Second law of thermodynamics; Infinite dimensional dynamical system; Dual variation

1. Introduction

In this paper, we study non-isothermal phase separation models of the Penrose–Fife type, which throughout this paper are denoted by $(\mathrm{P})_\mu := \{(1.1)\text{–}(1.6)\}$:

$$u_t + \lambda'(w)w_t - \Delta\alpha(u) = 0 \quad \text{a.e. in } Q := \Omega \times (0, +\infty), \tag{1.1}$$

$$w_t - \Delta\{\mu w_t - \kappa\Delta w + g(w) - \alpha(u)\lambda'(w)\} = 0 \quad \text{a.e. in } Q, \tag{1.2}$$

$$\nabla\alpha(u)\cdot\mathbf{n} = 0 \quad \text{a.e. on } \Sigma := \Gamma \times (0, +\infty), \tag{1.3}$$

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0362-546X/$ - see front matter
doi:10.1016/j.na.2007.01.015

$$\nabla w \cdot \mathbf{n} = \nabla\{\mu w_t - \kappa \Delta w + g(w) - \alpha(u)\lambda'(w)\} \cdot \mathbf{n} = 0 \quad \text{a.e. on } \Sigma, \tag{1.4}$$

$$u|_{t=0} = u_0 \quad \text{a.e. in } \Omega, \tag{1.5}$$

$$w|_{t=0} = w_0 \quad \text{a.e. in } \Omega, \tag{1.6}$$

where $\Omega \subset \mathbf{R}^N$ $(N = 1, 2, 3)$ is a bounded domain with a smooth boundary $\Gamma := \partial\Omega$; $\mathbf{n}$ is the outer unit normal vector on Γ; α is an increasing function from an open interval $(0, r^*)$ into $\mathbf{R}$ for some constant $r^* \in (1, +\infty]$; μ is a non-negative constant; κ is a positive constant; λ and g are sufficiently smooth functions from $\mathbf{R}$ into itself; u_0 and w_0 are prescribed initial data.

The original system of $(\mathrm{P})_\mu$ was proposed by Penrose and Fife in [9] to describe non-isothermal phase separation phenomena controlled by the absolute temperature; for example, non-isothermal spinodal decomposition of Ni (nickel)–Ti (titanium) binary alloy. From a physical point of view, u is a variable that only depends upon the absolute temperature of the material. w represents an order parameter that represents the ratio of the local concentration of one component. In this respect, the values taken by w are sometimes restricted in the interval $[0, 1]$ and $\bar{w} := 1 - w$ also represents the ratio of the local concentration of the other component. Namely, if the 0-area $\Omega_0 := \{x \in \Omega \mid w = 0\}$ is occupied by one component, the 1-area $\Omega_1 := \{x \in \Omega \mid w = 1\}$ is occupied by the other component. The area $\Omega_m := \Omega \setminus (\Omega_0 \cup \Omega_1) = \{x \in \Omega \mid 0 < w < 1\}$ is called the region of the mixture state of two components.

To describe the above setting, Kenmochi and Niezgódka [7] considered the maximal monotone graph β in $\mathbf{R} \times \mathbf{R}$ defined by

$$\beta(w) = \begin{cases} \emptyset, & \text{if } w > 1, \\ [0, +\infty), & \text{if } w = 1, \\ \{0\}, & \text{if } 0 < w < 1, \\ (-\infty, 0], & \text{if } w = 0, \\ \emptyset, & \text{if } w < 0, \end{cases}$$

and considered the following system {(1.1), (1.7)–(1.10)}, which is sometimes called the non-isothermal phase separation model with constraints, and is denoted by (PC) in this section:

$$w_t - \Delta\{\mu w_t - \kappa \Delta w + g(w) + \xi - \alpha(u)\lambda'(w)\} = f(x, t) \quad \text{a.e. in } Q, \tag{1.7}$$

$$\xi \in \beta(w) \quad \text{a.e. in } Q, \tag{1.8}$$

$$\nabla\alpha(u) \cdot \mathbf{n} + n_0\alpha(u) = h(x, t) \quad \text{a.e. on } \Sigma, \tag{1.9}$$

$$\nabla w \cdot \mathbf{n} = \nabla\{\mu w_t - \Delta w + g(w) + \xi - \alpha(u)\lambda'(w)\} \cdot \mathbf{n} = 0 \quad \text{a.e. on } \Sigma, \tag{1.10}$$

where n_0 is a positive constant. As can be seen from (1.9), in [7] the third type boundary condition is imposed on the nonlinear heat flux $\alpha(u)$.

On the other hand, [8] considered $(\mathrm{PC})_N$, which is the system {(1.1), (1.7), (1.8) and (1.10)}, imposing a homogeneous Neumann boundary condition for the heat flux $\alpha(u)$ (cf. (1.3)). Strictly speaking, in [8] they considered any maximal monotone graph α in $\mathbf{R} \times \mathbf{R}$ and (1.1), (1.3) and (1.4) hold for some $\tilde{\alpha}$, which is a measurable selection of $\alpha(u)$, because α is generally multi-valued. In other words, for some $\tilde{\alpha}$ satisfying the relation

$$\tilde{\alpha} \in \alpha(u) \quad \text{a.e. in } Q,$$

(1.1), (1.3) and (1.4), in which all $\alpha(u)$ are replaced by $\tilde{\alpha}$, are satisfied.

However, the common property among the systems $(\mathrm{P})_\mu$, (PC) and $(\mathrm{PC})_N$ is that the total mass of w is conserved in time, i.e.,

$$\frac{1}{|\Omega|}\int_\Omega w(t) = \frac{1}{|\Omega|}\int_\Omega w_0 =: m_0, \quad \forall t \geq 0. \tag{1.11}$$

Actually, it is derived from the kinetic equations of w (cf. (1.2) or (1.7)) and the boundary conditions (cf. (1.4) or (1.10)) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w(t) = 0, \quad \text{a.e. } t \geq 0.$$