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Viscosity methods of approximation for a common fixed point of a family of quasi-nonexpansive mappings

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Abstract

Let K be a nonempty closed convex subset of a real reflexive Banach space E that has weakly continuous duality mapping J_φ for some gauge φ. Let $T_i : K \to K, i = 1, 2, \ldots$, be a family of quasi-nonexpansive mappings with $F := \cap_{i \ge 1} F(T_i) \neq \emptyset$ which is a sunny nonexpansive retract of K with Q a nonexpansive retraction. For given $x_0 \in K$, let $\{x_n\}$ be generated by the algorithm $x_{n+1} := \alpha_n f(x_n) + (1 - \alpha_n)T_n(x_n), n \ge 0$, where $f : K \to K$ is a contraction mapping and $\{\alpha_n\} \subseteq (0, 1)$ a sequence satisfying certain conditions. Suppose that $\{x_n\}$ *satisfies condition* (A). Then it is proved that $\{x_n\}$ converges strongly to a common fixed point $\bar{x} = Qf(\bar{x})$ of a family $T_i, i = 1, 2, \ldots$. Moreover, $\bar{x}$ is the unique solution in F to a certain variational inequality.

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1. Introduction

Let E be a real Banach space with dual E^*. A gauge function is a continuous strictly increasing function $\varphi : \mathbf{R}^+ \to \mathbf{R}^+$ such that $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$. The duality mapping $J_\varphi : E \to E^*$ associated with a gauge function φ is defined by $J_\varphi(x) := \{u^* : \langle x, u^*\rangle = \|x\|.\|u^*\|, \|u^*\| = \varphi(\|x\|)\}, x \in E$, where $\langle ., .\rangle$ denotes the generalized duality pairing. In the particular case $\varphi(t) = t$, the duality map $J = J_\varphi$ is called the *normalized duality map*. We note that $J_\varphi(x) = \frac{\varphi(\|x\|)}{\|x\|} J(x)$. It is known that if E is smooth then J_φ is single valued and norm to w^* continuous (see, e.g., [6]).

Following Browder [3], we say that a Banach space E has the *weakly continuous duality mapping* if there exists a gauge function φ for which the duality map J_φ is single valued and weak to weak* sequentially continuous (i.e. if $\{x_n\}$ is a sequence in E weakly convergent to a point x, then the sequence $\{J_\varphi(x_n)\}$ converges weak* to $J_\varphi(x)$).

It is known that $l^p (1 < p < \infty)$ spaces have a weakly continuous duality mapping J_φ with a gauge $\varphi(t) = t^{p-1}$.

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Setting

$$\Phi(t) = \int_0^t \varphi(\tau)\mathrm{d}\tau, \quad t \geq 0, \tag{1.1}$$

one can see that $\Phi(t)$ is a convex function and $J_\varphi(x) = \partial\Phi(\|x\|)$, for $x \in E$, where ∂ denotes the subdifferential in the sense of convex analysis.

Let K be a nonempty closed convex subset of a real Banach space E. A mapping $T : K \to E$ is called *quasi-nonexpansive* if $\forall x \in K$ and $y \in F(T)$, the following inequality holds:

$$\|T(x) - y\| \leq \|x - y\|, \tag{1.2}$$

where $F(T) := \{x \in K : T(x) = x\} \neq \emptyset$. T is called *nonexpansive* if $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in K$. It is clear that a nonexpansive mapping T with $F(T) \neq \emptyset$ is quasi-nonexpansive. However, there exist quasi-nonexpansive mappings that are not nonexpansive. Let $T : \mathbf{R} \to \mathbf{R}$ be defined by $T(x) = \frac{x}{2}\sin\frac{1}{x}$ if $x \neq 0$ and $T0 = 0$. Then T is quasi-nonexpansive but not nonexpansive (see [7]). For a sequence $\{\alpha_n\}$ of real numbers in (0, 1) and an arbitrary $u \in K$, let the sequence $\{x_n\}$ in K be iteratively defined by $x_0 \in K$,

$$x_{n+1} := \alpha_{n+1}u + (1 - \alpha_{n+1})T(x_n), \quad n \geq 0, \tag{1.3}$$

where T is a nonexpansive mapping of K into itself.

Halpern [9] was the first to study the convergence of the algorithm (1.3) in the framework of Hilbert spaces. Lions [12] improved the result of Halpern, still in Hilbert spaces, by proving strong convergence of $\{x_n\}$ to a fixed point of T if the real sequence $\{\alpha_n\}$ satisfies the following conditions:

$$\text{(i)} \lim_{n\to\infty} \alpha_n = 0; \qquad \text{(ii)} \sum_{n=1}^{\infty} \alpha_n = \infty; \quad \text{and} \quad \text{(iii)} \lim_{n\to\infty} \frac{\alpha_n - \alpha_{n-1}}{\alpha_n^2} = 0. \tag{1.4}$$

It was noted that both Halpern's and Lions' conditions on the real sequence $\{\alpha_n\}$ excluded the natural choice, $\alpha_n := (n+1)^{-1}$. This was overcome by Wittmann [18] who proved that, still in Hilbert spaces, the strong convergence of $\{x_n\}$ if $\{\alpha_n\}$ satisfies the following conditions:

$$\text{(i)} \lim_{n\to\infty} \alpha_n = 0; \qquad \text{(ii)} \sum_{n=1}^{\infty} \alpha_n = \infty; \quad \text{and} \quad \text{(iii)}^* \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty. \tag{1.5}$$

Reich [15] extended the result of Wittmann to the class of Banach spaces which are uniformly smooth and have weakly sequentially continuous duality mappings.

In 2000, Moudafi [13] introduced viscosity approximation method and proved that if E is a real Hilbert space, for given $x_0 \in K$, the sequence $\{x_n\}$ generated by the algorithm

$$x_{n+1} := \alpha_n f(x_n) + (1 - \alpha_n)T(x_n), \quad n \geq 0, \tag{1.6}$$

where $f : K \to K$ is a contraction mapping with constant $\beta \in (0, 1)$ and $\{\alpha_n\} \subseteq (0, 1)$ satisfies certain conditions, converges strongly to a fixed point of T in K which is the unique solution to the following variational inequality:

$$\langle (I - f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in F(T).$$

Moudafi in [13] generalizes Browder's and Halpern's theorems in the direction of viscosity approximations. Viscosity approximations are very important because they are applied to convex optimization, linear programming, monotone inclusions and elliptic differential equations.

In 2004, Xu [20] studied further the viscosity approximation method for nonexpansive mappings in uniformly smooth Banach spaces. This result of Xu [20] extends Theorem 2.2 of Moudafi [13] to a Banach space setting. For details on the iterative methods, we refer the reader to [1].

Our concern now is the following: *Is it possible to construct a viscosity approximation sequence which converges to a common fixed point of a family of quasi-nonexpansive mappings in Banach spaces*?

Let K be closed and convex subset of a Banach space E. Let $T_i : K \to K, i = 1, \ldots,$ be a family of quasi-nonexpansive mappings with $F := \cap_{i\geq 1} F(T_i) \neq \emptyset$ which is a sunny nonexpansive retract of K and let $f : K \to K$