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A note on the geometry of Lie groups

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Abstract

The degenerate semi-Riemannian geometry of a Lie group is studied. Then a naturally reductive homogeneous semi-Riemannian space is obtained from the Lie group in a natural way.
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1. Introduction

If a Lie group G is semisimple then its Killing form is nondegenerate and it defines a bi-invariant semi-Riemannian metric on G. More generally, a Lie group G has the Levi decomposition

$$G = QM,$$

where M is a maximal semisimple Lie subgroup of G and Q is the radical of G (cf. [6]). Using these facts one can study semi-Riemannian geometry of a Lie group.

We particularly look at the case where the Killing form of G is degenerate to study the geometry of a Lie group G, and show that the degenerate distribution of the metric on G defined by the Killing form is integrable, and therefore G admits Levi-Civita connections with respect to this metric (cf. [1,3,7]). We also obtain the sectional curvature in terms of Lie invariants as in the semisimple case.

Finally, we assign a homogeneous space G/H to a Lie group G with Lie algebra $\mathfrak{g}$, where H is a Lie subgroup of G corresponding to the Lie subalgebra of $\mathfrak{g}$ defined by

$$\mathfrak{h} = \{X \in \mathfrak{g}; \langle X, Y\rangle = 0,\ \forall Y \in \mathfrak{g}\},$$

and prove that G/H is a naturally reductive homogeneous semi-Riemannian space for which the sectional curvature at a point is the same as the sectional curvature of G.

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2. Geometry of a Lie group from the metric defined by its Killing form

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Then the Killing form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbf{R}$ of G defined by $B(X, Y) = \text{trace}(ad_X \circ ad_Y)$ is a symmetric bilinear form on $\mathfrak{g}$. So B takes the form

$$B(X, X) = (x^1)^2 + \cdots + (x^p)^2 - (x^{p+1})^2 - \cdots - (x^{p+q})^2, \quad p + q \leq n,$$

for a suitable basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{g}$, where $X = \sum_{i=1}^{n} x^i X_i \in \mathfrak{g}$. Since B is left and $ad(G)$-invariant, it defines a bi-invariant (possibly degenerate) metric $\langle , \rangle$ on G such that

$$\langle , \rangle|_e = B.$$

If the signature of $\langle , \rangle$ is (p, q, r), $p + q + r = n$, then the constancy of p, q, r follows from the fact that G is connected and parallelizable.

Now we show that a degenerate vector field with respect to $\langle , \rangle$ is a Killing field.

Lemma 1. *Let G be a connected Lie group furnished with the metric $\langle , \rangle$ given by its Killing form of signature (p, q, r), $r > 0$. If we set*

$$TG^{\perp} = \bigcup_{g \in G} T_g G^{\perp}, \quad T_g G^{\perp} = \{X_g \in T_g G;\ \langle X_g, Y_g \rangle = 0, \forall Y_g \in T_g G\},$$

then a section X of $TG^{\perp}$ is a Killing field.

Proof. Let $X_1, \ldots, X_n$ be a basis of $\mathfrak{g}$ such that $X_1, \ldots, X_{p+q}$ is an orthonormal basis of a maximal nondegenerate subspace of $\mathfrak{g}$. If $X \in \Gamma(TG^{\perp})$, then we get

$$X\langle Y, Z \rangle = \sum_{k=p+q+1}^{n} \sum_{i,j=1}^{n} x^k \langle X_i, X_j \rangle (y^i X_k(z^j) + z^j X_k(y^i)),$$

where $X = \sum_{k=p+q+1}^{n} x^k X_k$, $Y = \sum_{i=1}^{n} y^i X_i$ and $Z = \sum_{j=1}^{n} z^j X_j$. Accordingly, we have that $\langle [X, Y], Z \rangle + \langle Y, [X, Z] \rangle$ is equal to

$$\sum_{k=p+q+1}^{n} \sum_{i,j=1}^{n} x^k (\langle X_i, X_j \rangle (y^i X_k(z^j) + z^j X_k(y^i)) + y^i z^j (\langle [X_k, X_i], X_j \rangle + \langle X_i, [X_k, X_j] \rangle)).$$

But, by bi-invariance, we have

$$\langle [X_k, X_i], X_j \rangle + \langle X_i, [X_k, X_j] \rangle = 0,$$

so the result follows. ■

This lemma shows that a connected Lie group G can be considered as a degenerate semi-Riemannian manifold; hence G has Levi-Civita connections which are all parametrized by $\Gamma(TG^{\perp})$-valued symmetric tensor fields of type $(1, 1)$ (cf. [1,3,7]).

For a degenerate semi-Riemannian manifold M with a Levi-Civita connection D, a curve γ is called a *geodesic* if $D_{\gamma'}\gamma' \in \Gamma(TM^{\perp})$. To be more explicit, we note that this definition reduces to the usual one in the semi-Riemannian case.

The following results are essentially the degenerate versions of those given in [5] (see Proposition 11.9 and Corollary 11.10) for which the metric is nondegenerate.

Proposition 2. *Let G be a connected Lie group with the metric $\langle , \rangle$ and let D be a Levi-Civita connection with respect to $\langle , \rangle$. Then:*

(i) *The inversion map $g \to g^{-1}$ is an isometry.*
(ii) $D_X Y = \frac{1}{2}[X, Y] + \mathcal{D}(X, Y)$ *for all $X, Y \in \mathfrak{g}$ and some symmetric smooth function $\mathcal{D} : \Gamma(TG) \times \Gamma(TG) \to \Gamma(TG^{\perp})$.*
(iii) *The geodesics of G starting at e are the one-parameter subgroups of G.*

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