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Critical points of a non-linear functional related to the one-dimensional Ginzburg–Landau model of a superconducting–normal–superconducting junction☆

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Abstract

A non-linear functional is studied, which is the limit of the one-dimensional Ginzburg–Landau model of a superconducting–normal–superconducting junction as the Ginzburg–Landau parameter tends to infinity. It is found that the functional may have one, two or three critical points according to various conditions of related parameters and the exact number of the critical points is obtained in each case. Moreover, the necessary and sufficient conditions for minimizers are also established.

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1. Introduction

The S–N–S junction (also called the Josephson junction) is very important in both the study of superconducting phenomena and the design of superconducting devices. It is a sample in which a thin layer of normal material is sandwiched between layers of superconducting materials. Let 2γ be the width of the sample and $2d$ be the width of the normal material with $d < \gamma$. By the Ginzburg–Landau theory, when the sample is subjected to a tangential magnetic field h, the order parameter ϕ and the induced magnetic potential a minimize the energy

$$E(\phi, a) = \int_{-\gamma}^{\gamma} \left\{2\phi^2 a^2 + \frac{2}{k^2}(\phi')^2 + 2(a' - h)^2\right\} \mathrm{d}x + \left\{\int_{-\gamma}^{-d} + \int_{d}^{\gamma}\right\} \phi^2(\phi^2 - 2)\mathrm{d}x + \int_{-d}^{d} 2\phi^2 \mathrm{d}x. \tag{1.1}$$

Here $\phi > 0$ and k is a positive constant known as the Ginzburg–Landau parameter (see [1–10] and the references therein). Since most superconducting materials with high critical temperature have high Ginzburg–Landau parameter

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k, it is very important to study the asymptotic behavior of the model as k tends to infinity. Let $\{\phi(x), a(x)\}$ be a minimizer of (1.1) in $H^1[-\gamma,\gamma]\times H^1[-\gamma,\gamma]$; then it is not difficult to see that, as $k\to+\infty$,

$$a \longrightarrow a_\infty, \quad \text{in } [-\gamma,\gamma],$$
$$\phi \longrightarrow \begin{cases} 0, & \text{in } [-d,d],\\ (\max\{1-a_\infty^2,0\})^{\frac{1}{2}}, & \text{in } [-\gamma,-d]\cup[d,\gamma], \end{cases}$$

where a_∞ is a minimizer of the following functional:

$$F(a) = 2\int_{-\gamma}^{\gamma}(a'-h)^2\mathrm{d}x - \left\{\int_{-\gamma}^{-d} + \int_{d}^{\gamma}\right\}[\max\{1-a^2,0\}]^2\mathrm{d}x \tag{1.2}$$

defined in $H^1[-\gamma,\gamma]$. In this paper, we shall only consider the special case where $a(x)$ is a odd function in $[-\gamma,\gamma]$. Define

$$H_0^1[0,\gamma] = \{a\in H^1[0,\gamma] \mid a(0)=0\}. \tag{1.3}$$

Then the functional (1.2) restricted to $H_0^1[0,\gamma]$ can be rewritten in the form

$$F(a) = 4\int_{0}^{\gamma}(a'-h)^2\mathrm{d}x - 2\int_{d}^{\gamma}[\max\{1-a^2,0\}]^2\mathrm{d}x \tag{1.4}$$

This paper is devoted to the study of the critical points of (1.4) in $H_0^1[0,\gamma]$, where d, γ and h are always positive constants, and $\gamma>d$. By a standard variational approach, a function $a(x)$ is a critical point of (1.4) in $H_0^1[0,\gamma]$ if and only if $a(x)\in C^2[0,d]\cup C^2[d,\gamma]\cup C^1[0,\gamma]$ and solves the following system:

$$\begin{cases} a'' = a\max\{1-a^2,0\}, & x\in[d,\gamma];\\ a''=0, & x\in[0,d];\\ a(0)=0, \quad a'(\gamma)=h; & \\ a\in C^1[0,\gamma] & \end{cases} \tag{1.5}$$

which, obviously, is equivalent to the following boundary problems:

$$a'' = a\max\{1-a^2,0\}, \quad x\in[d,\gamma]; \tag{1.6}$$

$$a'(d) = \frac{a(d)}{d}; \tag{1.7}$$

$$a'(\gamma) = h. \tag{1.8}$$

One of our main goals is to determine the number of solutions of (1.6)–(1.8) or, equivalently, (1.5). Before we state our main result on this aspect, we must give some basic ideas as follows.

It is easy to see that, if $a(x)$ be a solution of (1.6)–(1.8), then

$$a(x)>0, \qquad 0<a'(x)\le h, \qquad a''(x)\ge 0, \quad x\in[d,\gamma]. \tag{1.9}$$

Moreover, multiplying (1.6) and then integrating over $[x,y]\subset[d,\gamma]$, we find after some elementary calculations that, for any $x\in[d,\gamma]$ and any $y\in[d,\gamma]$,

$$2[a'(x)]^2 + [\max\{1-a^2(x),0\}]^2 = 2[a'(y)]^2 + [\max\{1-a^2(y),0\}]^2, \tag{1.10}$$

By (1.9) and (1.10),

$$\frac{a'(x)}{\sqrt{2[a'(d)]^2 + [\max\{1-a^2(d),0\}]^2 - [\max\{1-a^2(x),0\}]^2}} = \frac{1}{\sqrt{2}}.$$

Integrating the above identity over $x\in[d,\gamma]$ and using (1.7), we find

$$\int_{da'(d)}^{a(\gamma)} \frac{\mathrm{d}t}{\sqrt{2[a'(d)]^2 + [\max\{1-d^2[a'(d)]^2,0\}]^2 - [\max\{1-t^2,0\}]^2}} = \frac{\gamma-d}{\sqrt{2}}. \tag{1.11}$$