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First- and second-order discontinuous functional differential equations with impulses at fixed moments

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Abstract

We give sufficient conditions for the existence of extremal solutions to discontinuous and functional differential equations with impulses. Our main results are new even for ordinary differential equations without impulses.

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1. Introduction

We consider the impulsive functional boundary value problem

$$\begin{aligned} &x'(t) = f(t, x(t), x) \quad \text{for almost all (a.a.) } t \in J = [0, 1], \\ &I_k(x(t_k^+), x) = 0, \quad k = 1, 2, \ldots, p, \\ &B(x(0), x) = 0, \end{aligned} \tag{1.1}$$

where $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = 1$ is a fixed partition which corresponds to impulse effects.

Note that all of the elements in problem (1.1), namely the differential equation, the impulses and the boundary condition, depend functionally on the unknown, which means that the global

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form of the solution is involved at almost every moment and, in particular, in the determination of the impulses at the times t_k.

On the other hand the function B will be suitable to cover the usual conditions, such as initial or periodic. For the periodic boundary conditions it will suffice to define

$$B(u, \xi) = u - \xi(1) \quad \text{for } (u, \xi) \in \mathbb{R} \times X,$$

where X is a functional space where solutions will be looked for. Furthermore, not only the behavior of the solution at the boundary is involved in condition $B(x(0), x) = 0$ and, for instance, a condition of the form

$$x(0) = \int_{1/3}^{1/2} x(s)\mathrm{d}s,$$

can be studied in the frame of problem (1.1).

The special case of (1.1) corresponding to ordinary (nonfunctional) differential equations and explicit impulsive conditions depending simply on the values of the unknown on the instants t_k was studied in [4]. Here we are going to consider such general conditions over f that our result for (1.1) will be new even when applied to the particular case considered in [4].

For the simplicity and uniformity of notation we will put

$$t_0 = 0 \quad \text{and} \quad I_0 = B.$$

Since solutions x of (1.1) will satisfy $x(0^+) = x(0)$, we can rewrite our problem into the following form:

$$\begin{aligned} &x'(t) = f(t, x(t), x) \quad \text{for a.a. } t \in J = [0, 1],\\ &I_k(x(t_k^+), x) = 0, \quad k = 0, 1, \ldots, p. \end{aligned} \tag{1.2}$$

Let us note that we are going to treat the impulsive and the boundary conditions in the same way.

Finally, as an application of our result for (1.2), we will establish an existence result for

$$\begin{aligned} &u''(t) = h(t, u(t), u'(t)) \quad \text{for a.a. } t \in J = [0, 1],\\ &u(t_k^+) = \mu_k\, u(t_k) + \nu_k, \qquad u'(t_k^+) = \mathcal{I}_k(u'(t_k)), \quad k = 1, 2, \ldots, p,\\ &u(0) = A(u), \qquad B(u'(0), u') = 0, \end{aligned}$$

where, for $k = 1, 2, \ldots, p$, $\mu_k \geq 0$, $\nu_k \in \mathbb{R}$, $\mathcal{I}_k : \mathbb{R} \to \mathbb{R}$, $B : \mathbb{R} \times \Omega \to \mathbb{R}$, $A : \Omega \to \mathbb{R}$ is linear and nondecreasing, and Ω is a functional space to be defined later. This problem will be reduced to a problem of the type of (1.2) by order reduction. Similar reductions from ordinary to functional equations of lower order were developed by Cabada et al. [2,3] and Liz [8].

This paper is organized as follows. In Section 2 we include the relevant preliminary material such as definitions and technical lemmas; in Section 3 we prove our main result for (1.2), which is an existence result between given lower and upper solutions. In Section 4 we apply the previous result to prove another existence result for second-order ordinary differential equations with impulses and nonlinear boundary-functional conditions: to do that we use order reduction to a functional first-order problem. Finally we show the applicability of our theorems with some examples in Section 5.

Remark 1.1. The results of the present work remain valid, with obvious changes, if we replace the interval $J = [0, 1]$ by an arbitrary compact real interval $[a, b]$.