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Optimal decay rates for solutions of a nonlinear wave equation with localized nonlinear dissipation of unrestricted growth and critical exponent source terms under mixed boundary conditions

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Abstract

This article investigates optimal decay rates for solutions to a semilinear hyperbolic equation with *localized* interior damping and a source term. Both dissipation and the source are *fully nonlinear* and the growth rate of the source map may include critical exponents (for Sobolev's embedding $H^1 \to L^2$). Besides continuity and monotonicity, *no growth or regularity assumptions are imposed on the damping*. We analyze the system in the presence of Neumann-type boundary conditions including the *mixed* cases: Dirichlet–Neumann–Robin.

The damping affects a thin layer (a collar) near a portion of the boundary. To cope with the lack of control on the remaining section we develop a special method that accounts for propagation of the energy estimates from the dissipative region onto the entire domain. The Neumann system does not satisfy the Lopatinski condition in higher dimensions, hence the study of energy propagation in the absence of damping near the Neumann segment requires special geometric considerations.

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1. Introduction

1.1. The model

We consider a wave equation with nonlinear monotone damping, localized on a small (interior) portion of the spatial domain, and subject to Neumann condition on the boundary or its segment. We are interested in time-asymptotic behavior and related decay rates for the corresponding solutions evolving in the finite energy space $H^1 \times L^2$ (a precise definition of the space will be given later).

Let Ω be a smooth bounded connected domain in $\mathbb{R}^n$. The equation of interest is

$$\begin{cases} u_{tt} - \Delta u + \chi g(u_t) = f(u) & \text{in } Q_T \\ u(0) = u_0, \quad u_t(0) = u_1 & \text{in } \Omega \end{cases} \tag{1}$$

where χ is the characteristic function of a smooth connected subdomain $\Omega_\chi \subset \Omega$. We use the notation $Q_T \equiv (0, T) \times \Omega$, $\Sigma_T \equiv (0, T) \times \Gamma$. The forms $\|\cdot\|$ and $(\cdot, \cdot)_\Omega$ will denote respectively the norm and the inner product in $L^2(\Omega)$. The techniques we use permit us to analyze system (1) under a variety of boundary conditions. Before stating the available options for boundary dynamics let us address the geometrical properties of the domain.

1.2. Geometry of the domain

The process develops on a connected bounded open set $\Omega \subset \mathbb{R}^n$ with smooth boundary $\Gamma = \partial\Omega$. Divide Γ into two sections: the controlled segment Γ_C implicitly described by

$$\Omega_\chi = \{x \in \Omega \ : \ \text{dist}(x, \Gamma_C) < \eta\}, \quad \eta > 0$$

and the remaining (partially) unobserved section Γ_U. Both Γ_C, Γ_U are assumed to be connected and relatively open in Γ with $\Gamma = \Gamma_C \cup \Gamma_U$. Since Γ is also connected (except for the annular domain mentioned in Remark 2) we implicitly assume $\mu_{n-1}(\Gamma_C \cap \Gamma_U) > 0$, with μ_{n-1} denoting the $(n-1)$-dimensional Lebesgue measure. This overlap between Γ_C and Γ_U is important and will be used to construct a partition of Ω by smooth cutoff functions. Note that we place no restriction on the "thickness" η of the dissipative layer.

If Γ_U is subject to a Neumann-type boundary condition we require some geometrical assumptions to compensate the lack of control:

(AD-1) Let the curve Γ_U be given as a level set

$$\Gamma_U = \{y \in \mathbb{R}^n : \ell(y) = 0\}, \quad \ell \in C^3, \nabla\ell \neq 0 \text{ on } \Gamma_U$$

where ℓ is defined on a suitable domain in $\mathbb{R}^n$.

(AD-2) The Hessian matrix of ℓ is non-negative definite on Γ_U, which is a characterization for the surface $z = \ell(x)$ to be convex or having convex epigraph, or for the set Ω being a convex set near Γ_U (see [14, p. 302]), so that $\ell(x) \leq 0$ for $x \in \Omega$ near Γ_U, and $\nabla\ell$ points towards the exterior of Ω.

(AD-3) There exists a point $x_0 \in \mathbb{R}^n$, outside Ω, so that $(x - x_0) \cdot \nu(x) \leq 0$ on Γ_U with ν being the outward normal vector field on Γ (pointing in the same direction as $\nabla\ell(x)$).

(AD-4) In the case of a mixed Dirichlet–Neumann boundary, in order to cope with singularities propagating from junction J of the distinct boundary conditions (see definition (2) below), it is sufficient to assume that this interface falls into the controlled segment of the boundary: $J \subset \Gamma_C$.