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Fixed-point theorems in partially ordered metric spaces for operators with PPF dependence

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Abstract

In this paper, fixed point theorems of a nonlinear operator are developed, where the domain space $C[[a, b], E]$ is different from the range, E, which is a metric space with partial order. An application of the fixed point theorem to a periodic boundary value problem with delay is also given.

Keywords: Fixed point; Partially ordered set; PPF (past, present and future) dependence; Existence; Uniqueness; Lower solution; Periodic boundary value problem with delay

1. Introduction

The investigation of the existence of fixed points in partially ordered sets was first considered in [4]. This study was continued in [3] by assuming the existence of only a lower solution instead of the usual approach where both lower and upper solutions are assumed to exist. These fixed point theorems are then applied to obtain certain uniqueness and existence results for ordinary differential equations in [3]. A further extension of this idea was considered in [2].

A fixed point theory of nonlinear operators which are PPF (past, present and future) dependent in a Banach space was developed in [1]. The domain space of the nonlinear operator was taken as $C[[a, b], E]$ and the range space as E, a Banach space. An important example of such a nonlinear operator is a delay differential equation. The concept of Razumikhin class, $\Omega_0 = \{\phi \in E : \|\phi\|_{E_0} = \|\phi(c)\|_E\}$, for some $c \in [a, b]$ was utilized to obtain uniqueness of the fixed points, since the difference of two fixed points cannot lie in a Razumikhin class.

In this paper, a successful fusion of these two ideas is obtained in a partially ordered metric space. An application to a periodic boundary problem with delay is also given.

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2. Fixed-point theorems

Let $(E, \leq)$ be a partially ordered set and suppose that there exists a metric d such that (E, d) is a complete metric space. Let $E_0 = C[[a, b], E]$ and T be an operator defined from E_0 to E.

Theorem 2.1. *Assume*

(1) *T is continuous;*
(2) *T is monotonically nondecreasing;*
(3) *$d(T\phi, T\Psi) \leq \alpha d_0(\phi, \Psi), 0 \leq \alpha < 1, \phi \geq \Psi$ where $d_0(\phi, \Psi) = \max_{a \leq s \leq b} d(\phi(s), \Psi(s))$;*
(4) *There exists a lower solution ϕ_0 such that*

$$\phi_0(c) \leq T\phi_0 \quad \text{for some } c \in [a, b].$$

Then there exists a fixed point ϕ^* such that $\phi^*(c) = T\phi^*,\ c \in [a, b]$.

Proof. Let $T\phi_0 = x_1,\ x_1 \in E$. Choose $\phi_1 \in E_0$ such that $x_1 = \phi_1(c),\ c \in [a, b],\ \phi_1 \geq \phi_0,$ and $d(\phi_1(c), \phi_0(c)) = d_0(\phi_1, \phi_0)$. Then $T\phi_0 = \phi_1(c) \leq T\phi_1$, using the fact that T is monotonically nondecreasing.

Let $T\phi_1 = x_2,\ x_2 \in E$. Again choose $\phi_2 \in E_0$ such that $x_2 = \phi_2(c), \phi_2 \geq \phi_1$, and $d(\phi_2(c), \phi_1(c)) = d_0(\phi_2, \phi_1)$. Then $T\phi_0 = \phi_1(c) \leq T\phi_1 = \phi_2(c) \leq T\phi_2$. We obtain, by induction, for $n \in \mathbb{N}$

$$T\phi_0 = \phi_1(c) \leq T\phi_1 = \phi_2(c) \leq \cdots \leq T\phi_n = \phi_{n+1}(c) \leq T\phi_{n+1} \leq \ldots,$$
$$\phi_0 \leq \phi_1 \leq \cdots \leq \phi_n \leq \phi_{n+1} \leq \cdots \quad \text{and} \quad d(\phi_n(c), \phi_{n-1}(c)) = d_0(\phi_n, \phi_{n-1}).$$

Now, using induction, we can show that

$$d(T\phi_n, T\phi_{n-1}) \leq \alpha^n d_0(\phi_1, \phi_0), \quad \forall n \in \mathbb{N}. \tag{2.1}$$

For $n = 1$, since $\phi_0 \leq \phi_1$, from hypothesis (3) we have $d(T\phi_1, T\phi_0) \leq \alpha d_0(\phi_1, \phi_0)$.

Next, suppose that (2.1) is true for $n = k$, then
$d(T\phi_{k+1}, T\phi_k) \leq \alpha d_0(\phi_{k+1}, \phi_k) = \alpha d(\phi_{k+1}(c), \phi_k(c)) = \alpha d(T\phi_k, T\phi_{k-1}) \leq \alpha^{k+1} d_0(\phi_1, \phi_0)$, which completes the proof of (2.1).

We next show that $\{T\phi_n\}$ is a Cauchy sequence in E.

Let $m > n$, then

$$\begin{aligned} d(T\phi_m, T\phi_n) &\leq d(T\phi_m, T\phi_{m-1}) + d(T\phi_{m-1}, T\phi_{m-2}) + \cdots + d(T\phi_{n+1}, T\phi_n) \\ &\leq (\alpha^m + \alpha^{m-1} + \cdots + \alpha^{n+1}) d_0(\phi_1, \phi_0) \\ &= \alpha^{n+1}(\alpha^{m-(n+1)} + \cdots + \alpha + 1) d_0(\phi_1, \phi_0) \\ &= \alpha^{n+1} \frac{1 - \alpha^{m-n}}{1 - \alpha} d_0(\phi_1, \phi_0) \\ &< \frac{\alpha^{n+1}}{1 - \alpha} d_0(\phi_1, \phi_0). \end{aligned}$$

which implies that $\{T\phi_n\}$ is a Cauchy sequence in E.

Since E is a complete metric space and $\{\phi_n\}$ is a monotonically nondecreasing sequence, there exists a $\phi^* \in E_0$ such that $\phi_n \to \phi^*$ and $T\phi_n = \phi_{n+1}(c) \to \phi^*(c)$ as $n \to \infty$. To prove that ϕ^* is a fixed point of T, we first observe that since T is continuous on E_0, T is continuous at ϕ^*. Hence, given $\frac{\varepsilon}{2} > 0$, there exists a $\delta > 0$ such that $d(T\phi_{n+1}, T\phi^*) < \frac{\varepsilon}{2}$ whenever $d_0(\phi_{n+1}, \phi^*) < \delta$. Also, since $T\phi_n \to \phi^*(c)$, for any $\gamma = \min\{\frac{\varepsilon}{2}, \delta\}$, there exists $n_0 \in \mathbb{N}$ such that $d(T\phi_n, \phi^*(c)) < \gamma$ for $n \geq n_0$. Thus $d(T\phi^*, \phi^*(c)) \leq d(T\phi^*, T\phi_n) + d(T\phi_n, \phi^*(c)) < \frac{\varepsilon}{2} + \gamma < \varepsilon$.

Since ε is arbitrary, $T\phi^* = \phi^*(c)$. □

Theorem 2.2. *Assume that the conditions* (2), (3) *and* (4) *of Theorem* 2.1 *hold. Let E_0 be such that for any nondecreasing sequence $\{\phi_n\}$ converging to $\phi^* \in E_0$, $\phi_n \leq \phi^*$ for all n. Then T has a fixed point.*