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The inverse scattering problem for Schrödinger and Klein–Gordon equations with a nonlocal nonlinearity

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Abstract

We study the inverse scattering problem for the nonlinear Schrödinger equation and for the nonlinear Klein–Gordon equation with the generalized Hartree type nonlinearity. We reconstruct the nonlinearity from knowledge of the scattering operator, which improves the known results.

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1. Introduction

We consider the inverse scattering problem for the nonlinear Schrödinger equation

$$i\partial_t u + \Delta u = f(u) \tag{NLS}$$

and for the nonlinear Klein–Gordon equation

$$\partial_t^2 w - \Delta w + w = f(w) \tag{NLKG}$$

in space-time $\mathbb{R} \times \mathbb{R}^n$. Here, u is a complex-valued function of $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, w is either a complex-valued or real-valued function, $\partial_t = \partial/\partial t$, and Δ is the Laplacian in $\mathbb{R}^n$. The nonlocal nonlinear term $f(v)$ has the form

$$f(v) = \int_{\mathbb{R}^n} \mu(x, y)|v(x - y)|^2 v(x)\mathrm{d}y. \tag{1}$$

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In order to state the condition of μ, we define A^l_σ for $\sigma \in (0, n)$ and $l = 1, 2, \ldots$ as the set of all functions $\nu : [\mathbb{R}^n \setminus \{0\}]^2 \mapsto \mathbb{R}$ satisfying the following conditions:

(1) For $y \in \mathbb{R}^n \setminus \{0\}$, $\nu(\cdot, y) \in C^l(\mathbb{R}^n_x \setminus \{0\})$, and for $x \in \mathbb{R}^n \setminus \{0\}$, $\nu(x, \cdot)$ is measurable on $\mathbb{R}^n_y \setminus \{0\}$.

(2) For $(x, y) \in [\mathbb{R}^n \setminus \{0\}]^2$, $|\partial^\alpha_x \nu(x, y)| \le C_0 |y|^{-\sigma}$, where C_0 is independent of x, y and $0 \le |\alpha| \le l$.

(3) There exists $\lambda_0 \in C([\mathbb{R}^n \setminus \{0\}]^2)$ such that
- λ_0 is bounded on $[\mathbb{R}^n \setminus \{0\}]^2$
- λ_0 is not 0-function.
- λ_0 satisfies that either $\lambda_0 \ge 0$ or $\lambda_0 \le 0$.
- $\lim_{\alpha \downarrow 0} \nu(\alpha x, \alpha y)|\alpha y|^\sigma = \lambda_0(x, y)$ a.e.

Suppose that μ belongs to A^1_σ with unknown σ.

The nonlinear term $f(v)$ is the generalization of the Hartree term

$$G^\sigma_0(v) = \lambda(|\cdot|^{-\sigma} * |v|^2)v, \quad \lambda \in \mathbb{R} \setminus \{0\}.$$

The term G^σ_0 is an approximating expression of the nonlocal interaction of specific elementary particles. The equations (NLS) and (NLKG) with $f = G^\sigma_0$ are initially studied by Chadam and Glassey [3] and Menzala and Strauss [8], respectively. There is a substantial literature on the scattering theory for Hartree equations (see, for instance, [11] and references therein).

The inverse scattering problem for the nonlinear equation recovers the nonlinearity from the knowledge of the scattering operator. For the definition of the scattering operator for the nonlinear equation, see, e.g., Section 2 in [14]. As we will show later, under suitable conditions, the scattering operator is well-defined for (NLS) and for (NLKG).

The inverse scattering problem for the nonlinear Schrödinger equation with the Hartree term is initially studied by [17]. To introduce the other results, we define the following two terms:

$$G_1(v) = (\lambda_1(\cdot)|\cdot|^{-\sigma} * |v|^2)v,$$
$$G_2(v) = \lambda_2(x)(|\cdot|^{-\sigma} * |v|^2)v,$$

where $\lambda_j \in C^1(\mathbb{R}^n) \cap W^1_\infty(\mathbb{R}^n)$, $\lambda_j(0) \ne 0$, $j = 1, 2$. We remark that the terms G_1 and G_2 satisfy (1) with $\mu = \lambda_1(y)|y|^{-\sigma} \in A^1_\sigma$ and $\mu = \lambda_2(x)|y|^{-\sigma} \in A^1_\sigma$, respectively. Watanabe [19] determined σ of the term G_1 if λ_1 is a constant. However, the method of [19] is not applicable in the case where λ_1 is not constant. It was suggested in [19] that we can easily reconstruct the term λ_2 if σ is a given number. Thus, if we can make a formula for determining σ of G_2, then we can reconstruct G_2. However, the method for determining σ of G_2 is not known in the case where λ_2 is not a constant.

The inverse scattering problem for the nonlinear Klein–Gordon equation with the Hartree term is initially studied by [14], which proved the uniqueness on identifying $\mu = \mu(y)$. We remark that, for $\lambda > 0$ with $\lambda \ne 1$, and for any nontrivial solution of a free Klein–Gordon equation $\phi(t, x)$, a rescaled function $\phi(\lambda t, \lambda x)$ does not solve the equation. As a result, the method for making the reconstruction formula for G^σ_0 in (NLS) is not applicable to the same term in (NLKG).

Our aim in this paper is to give an almost complete answer to the above problem, which has not previously been shown. More precisely, for (NLS) and (NLKG), and for $j = 1, 2$, we shall determine the σ of G_j even if λ_j is not constant. Since f with $\mu \in A^1_\sigma$ is a generalization of G_j, it is sufficient to make the formula for determining σ of $\mu \in A^1_\sigma$.