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Wasserstein kernels for one-dimensional diffusion problems

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Abstract

We treat the evolution as a gradient flow with respect to the Wasserstein distance on a special manifold and construct the weak solution for the initial-value problem by using a time-discretized implicit scheme. The concept of *Wasserstein kernel* associated with one-dimensional diffusion problems with Neumann boundary conditions is introduced. On the basis of this, features of the initial data are shown to propagate to the weak solution at almost all time levels, whereas, in a case of interest, these features even help with obtaining the weak solution. Numerical simulations support our theoretical results.

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1. Introduction

1.1. Overview

This work is organized as follows. First we present the general problem and the main tools pertaining to the present approach. We define the Wasserstein kernel, then we use it in two different instances to prove convergence of the time interpolants (based on the minimizers from the implicit schemes) to the weak solution. Convergence for an inhomogeneous porous medium equation with exponent $\gamma = 2$ is analyzed first, and this is followed by a much more detailed study of a Stefan problem. The latter includes numerical simulations that confirm the theoretical results.

Let us begin by recalling the setting in [15]. Basically, there we study the N-dimensional generalization of the following inhomogeneous diffusion problem:

$$u_t - f(u)_{xx} = g(x, t, u) \quad \text{in } (0, 1) \times (0, T) \quad \text{and} \quad f(u)_x = 0 \quad \text{on } \{0, 1\} \times (0, T), \tag{P_f}$$

where $f : [0, \infty) \to \mathbb{R}$, $g : \mathbb{R}^3 \to \mathbb{R}$ are functions with certain properties (see [15]; the cases studied here are encompassed). Let $0 < T < \infty$. We recall the following:

Definition 1. A function $u \in L^\infty((0, 1) \times (0, T))$ is a weak solution for (P_f) if it satisfies $f(u)_x \in L^1((0, 1) \times (0, T))$ and

$$\int_0^T \int_0^1 \{u\zeta_t - f(u)_x\zeta_x + g(\cdot, \cdot, u)\zeta\} \mathrm{d}x\mathrm{d}t = -\int_0^1 u_0\zeta(\cdot, 0)\mathrm{d}x,$$

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for all $\zeta \in C_c^\infty([0,1] \times [0,T))$.

We also have a weaker notion, given by:

Definition 2. A function $u \in L^\infty((0,1) \times (0,T))$ is a generalized solution for (P_f) if it satisfies

$$\int_0^T \int_0^1 \{u\zeta_t + f(u)\zeta_{xx} + g(\cdot,\cdot,u)\zeta\}\mathrm{d}x\mathrm{d}t = -\int_0^1 u_0\zeta(\cdot,0)\mathrm{d}x,$$

for all $\zeta \in C_c^\infty([0,1] \times [0,T))$ such that $\zeta_x(0,t) = \zeta_x(1,t) = 0$ for all $t > 0$.

Consider now two Lebesgue integrable nonnegative functions u_1 and u_2 of the same positive total mass. We recall the definition of the Wasserstein distance of order 2 as found in [7,14,12], etc. For this purpose we introduce

$$P(u_1,u_2) := \left\{ \text{nonnegative Borel measure } \mu \text{ on } [0,1]\times[0,1] \,\middle|\, \int_0^1\int_0^1 \xi(x)\mathrm{d}\mu(x,y) = \int_0^1 \xi(x)u_1(x)\mathrm{d}x \text{ and } \int_0^1\int_0^1 \xi(y)\mathrm{d}\mu(x,y) = \int_0^1 \xi(y)u_2(y)\mathrm{d}y \text{ for all } \xi \in C[0,1] \right\}$$

as the set of all admissible transfer plans, between the nonnegative finite measures (of the same total mass) $u_1\mathrm{d}x$ and $u_2\mathrm{d}x$.

Definition 3. The (square of the) Wasserstein distance (of order 2) is defined as

$$d(u_1,u_2)^2 := \inf_{\mu \in P(u_1,u_2)} \int_0^1\int_0^1 |x-y|^2\mathrm{d}\mu(x,y).$$

Properties of the Wasserstein distance will be referenced as they are used throughout this paper.

As Kinderlehrer and Walkington discuss in [10], the functional

$$S_f(u) := \int_0^1 \Phi_f(u)\mathrm{d}x, \quad u \in \mathcal{M}_{u^*} \text{ (see (1.3))},$$

is decreasing along the trajectories of (P_f), where Φ_f satisfies $y\Phi_f'(y) - \Phi_f(y) = f(y)$, but one cannot realize u as the *gradient flow* of the functional S_f in a conventional sense. We demonstrate in [15] that, formally, a solution for (P_f) is a gradient flow of S_f on a certain manifold with respect to the Wasserstein distance. This is done by means of the equivalence between the Wasserstein distance and a certain induced distance on $\mathcal{M}_{u^*}$ (proved by Otto in [13]). The approximants for the weak solution are obtained by time-step discretizing the gradient flow. Next, we go briefly over this construction.

1.2. Preliminaries

Let $u^* \in L^1(0,1)$ be nonnegative of positive total mass and let $h > 0$ be fixed. We define the nonlinear functional $F[h,u^*] : \mathcal{M}_{u^*} \to [0,\infty)$ by

$$F[h,u^*](u) = \frac{1}{2h}d(u,u^*)^2 + S_f(u). \tag{1.1}$$

The gradient flow of S_f on $\mathcal{M}_{u^*}$ w.r.t. the Wasserstein distance admits a time-step discretization of the form (see [13, 12])

$$\text{Minimize } F[h,u^*](u) \text{ among all } u \in \mathcal{M}_{u^*}, \tag{1.2}$$

where

$$\mathcal{M}_{u^*} := \left\{ u : (0,1) \to [0,\infty) \,\middle|\, u \text{ is measurable and } \int_0^1 u\mathrm{d}x = \int_0^1 u^*\mathrm{d}x \right\}. \tag{1.3}$$

We will now state the following: