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ScienceDirect
Nonlinear Analysis 67 (2007) 2623–2631
Nonlinear Analysis
www.elsevier.com/locate/na

The global attractor of a competitor–competitor–mutualist reaction–diffusion system with time delays

C.V. Pao*

Department of Mathematics, North Carolina State University, Raleigh, NC 27695-8205, United States

Received 4 May 2006; accepted 11 September 2006

Abstract

The aim of this paper is to investigate the asymptotic behavior of time-dependent solutions of a three-species reaction–diffusion system in a bounded domain under a Neumann boundary condition. The system governs the population densities of a competitor, a competitor–mutualist and a mutualist, and time delays may appear in the reaction mechanism. It is shown, under a very simple condition on the reaction rates, that the reaction–diffusion system has a unique constant positive steady-state solution, and for any nontrivial nonnegative initial function the corresponding time-dependent solution converges to the positive steady-state solution. An immediate consequence of this global attraction property is that the trivial solution and all forms of semitrivial solutions are unstable. Moreover, the state–state problem has no nonuniform positive solution despite possible spatial dependence of the reaction and diffusion. All the conclusions for the time-delayed system are directly applicable to the system without time delays and to the corresponding ordinary differential system with or without time delays.

MSC: 35K57; 35K50; 34K20

Keywords: Reaction–diffusion system; Asymptotic behavior; Positive solutions; Global attraction; Time delays; Upper and lower solutions

1. Introduction

Large time behavior of solutions of reaction–diffusion systems is one of the most important concerns in population dynamics. Since many such equations possess multiple steady-state solutions, the determination of the asymptotic behavior of the time-dependent solution in relation to the steady-state solutions becomes rather delicate, especially in determining the exact limit of the time-dependent solution. In this paper we investigate the stability problem for a special reaction–diffusion system, called a competitor–competitor–mutualist system, which governs the population densities of a competitor–mutualist u, a competitor v, and a mutualist w in a bounded domain. The mathematical problem, including possible time delays in the reaction mechanism, is given by

$$u_t - L_1 u = \beta_1(x)u[a_1 - b_1 u - (c_1' v + c_1'' v_{\tau_2})/(1 + \sigma_1' w + \sigma_1'' w_{\tau_3})]$$
$$v_t - L_2 v = \beta_2(x)v[a_2 - (b_2' u + b_2'' u_{\tau_1}) - c_2 v]$$

* Tel.: +1 919 515 2382; fax: +1 919 515 3798.
E-mail address: cvpao@math.ncsu.edu.

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doi:10.1016/j.na.2006.09.027

$$\begin{aligned} &w_t - L_3 w = \beta_3(x) w[a_3 - b_3 w/(1+\sigma_2' u + \sigma_2'' u_{\tau_1})] \quad (t>0, x\in\Omega) \\ &\partial u/\partial\nu = \partial v/\partial\nu = \partial w/\partial\nu = 0 \quad (t>0, x\in\partial\Omega) \\ &u(t,x)=\eta_1(t,x) \quad (t\in I_1), \qquad v(t,x)=\eta_2(t,x) \quad (t\in I_2), \qquad w(t,x)=\eta_3(t,x) \quad (t\in I_3),\ (x\in\Omega) \end{aligned} \tag{1.1}$$

where $u_{\tau_1} \equiv u(t-\tau_1, x)$, $v_{\tau_2} = v(t-\tau_2, x)$, $w_{\tau_3} = w(t-\tau_3, x)$ with time delays $\tau_i > 0, i = 1, 2, 3$, Ω is a smooth bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega$, and $\partial/\partial\nu$ denotes the outward normal derivative on $\partial\Omega$. For each $i = 1, 2, 3$, I_i is the interval $[-\tau_i, 0]$, a_i, b_i, c_i and τ_i, where $b_2 = b_2' + b_2''$, $c_1 = c_1' + c_1''$ are positive constants, c_1', c_1'', b_2', b_2'', and σ_j', σ_j'', $j = 1, 2$, are all nonnegative constants, and L_i is a uniformly elliptic operator in the form

$$L_i \equiv \sum_{j,k=1}^{n} a_{j,k}^{(i)}(x) \frac{\partial^2}{\partial x_j \partial x_k} + \sum_{j=1}^{n} b_j^{(i)}(x) \frac{\partial}{\partial x_j}.$$

Of special interest is the diffusion–convection operator

$$L_i = D^{(i)}(x)\nabla^2 + \mathbf{c}^{(i)}(x)\cdot\nabla \quad (i = 1, 2, 3)$$

where ∇ and ∇^2 are the gradient and Laplacian operators in Ω, $D^{(i)}(x) \ge d^{(i)} > 0$ in Ω, and $\mathbf{c}^{(i)}(x) = (c_1^{(i)}(x), \ldots, c_n^{(i)}(x))$. It is assumed that the coefficients of L_i and the functions β_i and η_i are smooth functions in their respective domains, $\beta_i(x) > 0$ in Ω, and $\eta_i(t, x) \ge 0$ in $I_i \times \overline{\Omega}$, where $\overline{\Omega} = \Omega \cup \partial\Omega$ (see [8,10] for some detailed assumptions). The constants c_1' or c_1'', b_2' or b_2'' as well as σ_j' or σ_j'', $j = 1, 2$, are allowed to be zero. In particular, if $c_1'' = b_2'' = \sigma_1'' = \sigma_2'' = 0$, then problem (1.1) is reduced to the standard competitor–competitor–mutualist system without time delays (see (1.5) below).

To investigate the asymptotic behavior of the solution of (1.1) we also consider the corresponding steady-state system

$$\begin{aligned} &-L_1 u = \beta_1(x) u[a_1 - b_1 u - c_1 v/(1+\sigma_1 w)] \\ &-L_2 v = \beta_2(x) v[a_2 - b_2 u - c_2 v] \\ &-L_3 w = \beta_3(x) w[a_3 - b_3 w/(1+\sigma_2 u)] \quad (x\in\Omega) \\ &\partial u/\partial\nu = \partial v/\partial\nu = \partial w/\partial\nu = 0 \quad (x\in\partial\Omega) \end{aligned} \tag{1.2}$$

where

$$c_1 = c_1' + c_1'', \qquad b_2 = b_2' + b_2'', \qquad \sigma_1 = \sigma_1' + \sigma_1'', \qquad \sigma_2 = \sigma_2' + \sigma_2''. \tag{1.3}$$

It is clear that problem (1.2) has the trivial solution (0, 0, 0) and various forms of semitrivial solutions, including $(a_1/b_1, 0, 0)$, $(0, a_2/c_2, 0)$, $(0, 0, a_3/b_3)$ and some others (see [13,16]).

The competitor–competitor–mutualist model was initiated by Rai, Freedman and Addicott [13] for the ordinary differential system

$$\begin{aligned} &u_t = u[a_1 - b_1 u - (c_1' v + c_1'' v_{\tau_2})/(1+\sigma_1' w + \sigma_1'' w_{\tau_3})] \\ &v_t = v[a_2 - (b_2' u + b_2'' u_{\tau_1}) - c_2 v] \\ &w_t = w[a_3 - b_3 w/(1+\sigma_2' u + \sigma_2'' u_{\tau_1})] \quad (t>0) \\ &u(t)=\eta_1(t) \quad (t\in I_1), \qquad v(t)=\eta_2(t) \quad (t\in I_2), \qquad w(t)=\eta_3(t) \quad (t\in I_3), \end{aligned} \tag{1.4}$$

without time delays (that is, $c_1'' = b_2'' = \sigma_1'' = \sigma_2'' = 0$). They obtain conditions for the boundedness of the global solution and local stability or instability of the various equilibria. Their model was extended by Zheng [16] to the reaction–diffusion system (without time delays)

$$\begin{aligned} &u_t - D_1\nabla^2 u = u[a_1 - b_1 u - c_1 v/(1+\sigma_1 w)] \\ &v_t - D_2\nabla^2 v = v[a_2 - b_2 u - c_2 v] \\ &w_t - D_3\nabla^2 w = w[a_3 - b_3 w/(1+\sigma_2 u)] \quad (t>0, x\in\Omega) \\ &\partial u/\partial\nu = \partial v/\partial\nu = \partial w/\partial\nu = 0, \quad (t>0, x\in\partial\Omega) \\ &u(0,x)=\eta_1(x), \qquad v(0,x)=\eta_2(x), \qquad w(0,x)=\eta_3(x) \quad (x\in\Omega). \end{aligned} \tag{1.5}$$