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Euler equations and approximations for the minimizers of Heisenberg target[☆]

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Abstract

In this paper, we study the Euler equations and derive approximations of the minimizers for a Heisenberg group target. There are some techniques in the arguments for proving the results. This is in order to overcome the obstacles which are due to the nonlinear structure of the group laws.

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1. Introduction

We firstly recall that the Heisenberg group $\mathbf{H}^n$ (see [1]) is the Lie group whose underlying manifold is $\mathbf{C}^n \times \mathbf{R}$, $n \in \mathbf{N}$, endowed with the group law algebra $\mathbf{g} = \mathbf{R}^{2n+1}$, and with a nonabelian group law:

$$(z, t) \cdot (z', t') = (z + z', t + t' + 2Im z \cdot z'), \tag{1.1}$$

where for $z, z' \in \mathbf{C^n}$ we have $z \cdot z' = \sum_{j=1}^{n} z_j \bar{z}'_j$. Setting $z_j = x_j + \mathrm{i} y_j$, $(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)$ forms a real coordinate system for $\mathbf{H}^n$. In this coordinate system the vector fields

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad j = 1, 2, \ldots, n, \tag{1.2}$$

and $T = \frac{\partial}{\partial t}$ generate the real Lie algebra of left-invariant vector fields on $\mathbf{H}^n$. It is easy to check that $[X_j, Y_k] = -4\delta_{jk} \frac{\partial}{\partial t}$, $j, k = 1, 2, \ldots, n$, and that all other commutators are trivial. Furthermore, for the group, there is a natural dilation defined by

$$\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t), \quad \lambda > 0, \tag{1.3}$$

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and a metric $d(u, v)$ defined by (see [2])

$$d(u, v) = |vu^{-1}| = [((x_v - x_u)^2 + (y_v - y_u)^2)^2 + (t_v - t_u + 2(x_u y_v - x_u y_v))^2]^{\frac{1}{4}}. \tag{1.4}$$

In particular, a homogeneous gauge $|u|_{\mathbf{H}^n}$ is defined as

$$[(x^2 + y^2)^2 + t^2]^{\frac{1}{4}} = (|z|^4 + t^2)^{\frac{1}{4}}.$$

We see that $\mathbf{H}^n$ possesses a nonlinear structure about the group law, which is one of the differences between $\mathbf{H}^n$ and a general Riemann manifold. The fact that $\mathbf{H}^n$ is a singular space can be intuitively understood also in the light of a recent result of Christodoulou (see [1]) who proved that the Heisenberg group can be constructed as the continuum limit of a crystalline material.

Concerning the minimizer maps with Heisenberg group target, existence and Lipschitz continuity have been proved by Capogna and Lin [3]. However, we cannot get any higher order or higher dimensional regularity about the maps as there are not Euler equations and the related iterative techniques as in the classical case. These problems must be very difficult and complicated! In this paper, our goals are to study the Euler equations of the minimizers and to derive an approximation for the minimizers. We think that the equations and the approximation are significant for discussing further the higher order or higher dimensional regularities of the minimizers.

2. Preliminary results

In this section, we give some definitions and Lemmas related to the Sobolev type space $W^{1,p}(\Omega, \mathbf{H}^n)$ and the minimizer maps. The partial results come from the ideal of Capogna and Lin (see [3]).

Definition 2.1 (*[3]*). Let $1 < p < \infty$, Ω be a bounded domain in $\mathbf{R^m}$. A function $u = (z, t) : \Omega \to \mathbf{H}^n$ is in $L^p(\Omega, \mathbf{H}^n)$ if for some $h_0 \in \Omega$, one has

$$\int_\Omega (d(u(h), u(h_0)))^p \mathrm{d}h < \infty.$$

A function $u = (z, t) : \Omega \to \mathbf{H}^n$ is in a Sobolev type space $W^{1,p}(\Omega, \mathbf{H}^n)$ if $u \in L^p(\Omega, \mathbf{H}^n)$ and

$$E_{p,\Omega}(u) = \sup_{f \in C_c(\Omega), 0 \le f \le 1} \limsup_{\epsilon \to 0} \int_\Omega f(h) e_{u,\epsilon}(h) \mathrm{d}h < \infty,$$

where $e_{u,\epsilon}(h) = \int_{|h-q|=\epsilon} \left(\frac{d(u(h),u(q))}{\epsilon}\right)^p \frac{d\sigma_\epsilon(q)}{\epsilon^{m-1}}$. $E_{p,\Omega}(u)$ is called a p-energy of u on Ω.

Lemma 2.1. *If $u = (x, y, t) = (z, t) \in W^{1,p}(\Omega, \mathbf{H}^n)$, then*

$$\nabla t = 2(y \nabla x - x \nabla y), \tag{2.1}$$

and $\nabla t \in L^{p/2}(\Omega)$.

Lemma 2.2 (*Poincaré Type Inequality for the Heisenberg Group Target*). *Suppose Ω is a bounded and connected Lipschitz domain in $\mathbf{R^m}$. Let $p \ge 2$. Then there exists a constant C depending only on Ω, m and p, such that for every function $u = (x, y, t) = (z, t) \in W^{1,p}(\Omega, \mathbf{H}^n)$, it holds that*

$$\int_\Omega (d(u(h), \lambda_u))^p \mathrm{d}q \le C_\Omega E_{p,\Omega}(u) = C_\Omega \int_\Omega |\nabla z|^p(h) \mathrm{d}h. \tag{2.2}$$

Here $\lambda_u = (\lambda_x, \lambda_y, \lambda_t)$, $\lambda_f = \frac{1}{|\Omega|} \int_\Omega f(h) \mathrm{d}h$ and $\nabla z = (\nabla x, \nabla y)$.

Proof. Obviously, $\lambda_u \in W^{1,p}(\Omega, \mathbf{H}^n)$. Using a C_p-inequality [8], we have

$$(d(u, \lambda_u))^p \le C_p \left[|x - \lambda_x|^p + |y - \lambda_y|^p + |t - \lambda_t + 2(\lambda_x y - \lambda_y x)|^{\frac{p}{2}}\right].$$