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Blow-up and global solutions for nonlinear reaction–diffusion equations with Neumann boundary conditions

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Abstract

The type of problem under consideration is

$$\begin{cases} \left((1+u)\ln^{\alpha}(1+u)\right)_t = \nabla\cdot\left(\ln^{\sigma}(1+u)\nabla u\right) + (1+u)\ln^{\beta}(1+u), & \text{in } D\times(0,T),\\ \dfrac{\partial u}{\partial n} = 0, & \text{on } \partial D\times(0,T),\\ u(x,0) = u_0(x) > 0, & \text{in } \bar{D}, \end{cases}$$

where $D \subset R^N$ is a bounded domain with smooth boundary ∂D, $N \geq 2$. It is proved that if $\beta - 1 > \sigma \geq \alpha \geq 0$, the positive solution $u(x,t)$ blows up globally in $\bar{D}$, whereas if $0 \leq \beta \leq \sigma \leq \alpha - 1$, the positive solution $u(x,t)$ is global solution. Furthermore, an upper bound of the "blow-up time", an upper estimate of the "blow-up rate", and an upper estimate of the global solutions are given.

MSC: 35K57; 35K55; 35K10

Keywords: Nonlinear reaction–diffusion equations; Blow-up solutions; Global solutions

1. Introduction

Blow-up solutions and global solutions for nonlinear reaction–diffusion equations reflect instability and stability of heat and mass transport processes respectively. Papers [3,5] dealt with the Cauchy problem:

$$\begin{cases} u_t = \Delta u + (1+u)\ln^{\beta}(1+u), & \text{in } R^N\times(0,T),\\ u(x,0) = u_0(x) \geq 0, & \text{in } R^N, \end{cases}$$

where $N \geq 1$. Papers [5,6,8] researched the initial and boundary value problem:

$$\begin{cases} u_t = \Delta u + (1+u)\ln^{\beta}(1+u), & \text{in } D\times(0,T),\\ u = 0, & \text{on } \partial D\times(0,T),\\ u(x,0) = u_0(x) \geq 0, & \text{in } \bar{D}, \end{cases}$$

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where $\bar{D}$ is closure of D. Papers [2,4,5] and the book [10] discussed the initial and boundary value problem:

$$\begin{cases} u_t = \nabla \cdot \left(\ln^{\sigma}(1+u)\nabla u\right) + (1+u)\ln^{\beta}(1+u), & \text{in } D \times (0,T), \\ u = 0, & \text{on } \partial D \times (0,T), \\ u(x,0) = u_0(x) \geq 0, & \text{in } \bar{D}, \end{cases}$$

where ∇ is gradient operator. In this paper, we shall study the initial and boundary value problem:

$$\begin{cases} ((1+u)\ln^{\alpha}(1+u))_t = \nabla \cdot (\ln^{\sigma}(1+u)\nabla u) + (1+u)\ln^{\beta}(1+u), & \text{in } D \times (0,T), & \text{(a)} \\ \dfrac{\partial u}{\partial n} = 0, & \text{on } \partial D \times (0,T), & \text{(b)} \\ u(x,0) = u_0(x) > 0, & \text{in } \bar{D}, & \text{(c)} \end{cases} \tag{1.1}$$

where $D \subset R^N$ is a bounded domain with smooth boundary ∂D, $N \geq 2$, $\partial/\partial n$ represents the outward normal derivative on ∂D, $u_0(x)$ satisfies the compatibility conditions and T is the maximum existence time of $u(x,t)$. The nonlinearities of (1.1)(a)–(c) consist of nonlinear reaction, nonlinear diffusion and nonlinear convection, described by $(1+u)\ln^{\beta}(1+u)$, $(1+u)\ln^{\alpha}(1+u)$ and $\ln^{\sigma}(1+u)$, respectively. We wish to know what interactions among the three nonlinear mechanisms result in the blow-up positive solutions and global positive solutions of (1.1)(a)–(c). In the present paper, it is proved that if $\beta - 1 > \sigma > \alpha \geq 0$, the positive solution $u(x,t)$ of (1.1)(a)–(c) blows up globally in $\bar{D}$, whereas if $0 \leq \beta \leq \sigma \leq \alpha - 1$, the positive solution $u(x,t)$ of (1.1)(a)–(c) is a global solution. Furthermore, an upper bound of the "blow-up time", an upper estimate of the "blow-up rate", and an upper estimate of the global solutions are given. The upper estimates of blow-up rate are optimal and could lead to stabilization to Hamilton–Jacobi similarity solutions (see [7]). Our approach depends heavily upon constructing auxiliary functions and using maximum principles.

The content of this paper is organized as follows. In Section 2 we shall study the blow-up solutions of (1.1)(a)–(c). In Section 3 we shall research the global solutions of (1.1)(a)–(c).

2. Blow-up solutions

Our main result is the following theorem:

Theorem 1. *Let $u(x,t)$ be a $C^3(D \times (0,T)) \cap C^2(\bar{D} \times [0,T))$ positive solution of (1.1)(a)–(c). Assume $\beta - 1 > \sigma \geq \alpha \geq 0$ and*

$$c = \min_{\bar{D}} \left\{ \frac{\ln^{\sigma-\beta-\alpha+1}(1+u_0)}{(1+u_0)[\ln(1+u_0)+\alpha]} \left[\nabla \cdot \left(\ln^{\sigma}(1+u_0)\nabla u_0\right) + (1+u_0)\ln^{\beta}(1+u_0)\right] \right\} > 0. \tag{2.1}$$

Then $u(x,t)$ blows up globally in $\bar{D}$ and "blow-up time"

$$T \leq \frac{\ln^{\sigma-\beta+1}(1+M_0)}{c(\beta-\sigma-1)} \tag{2.2}$$

as well as

$$u(x,t) \leq \mathrm{e}^{[c(\beta-\sigma-1)(T-t)]^{\frac{1}{\sigma-\beta+1}}} - 1, \tag{2.3}$$

where $M_0 = \max_{\bar{D}} u_0(x)$.

Proof. Consider the auxiliary function

$$\Phi = -\ln^{\sigma}(1+u)u_t + c(1+u)\ln^{\beta}(1+u), \tag{2.4}$$

from which we find

$$\nabla\Phi = -\frac{\sigma}{1+u}\ln^{\sigma-1}(1+u)u_t\nabla u - \ln^{\sigma}(1+u)\nabla u_t + \left[c\ln^{\beta}(1+u) + c\beta\ln^{\beta-1}(1+u)\right]\nabla u, \tag{2.5}$$